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Measurability of Gravitational Field Strengths (*).

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1. – Introduction.

The question of measurability of the electromagnetic field strengths was first raised by LANDAU and PEIERLS [1] who concluded that the measurability of electromagnetic field strengths is subject to limitations which go beyond the assumptions of quantum electrodynamics, and that quantum electrodynamics gets away from its physical foundations. BOHR and ROSENFELD [2] in their eminent work pointed out that Landau and Peierl's result does not necessarily affect the consistency of the theory because in studying the measurability of electromagnetic field strengths one has to use test bodies of finite

(*) A thesis submitted to the Faculty of the University of North Carolina in May, 1960 in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Physics.

(1) L. LANDAU and R. PEIERS: *Zeits. f. Phys.*, **69**, 56 (1931).

(2) N. BOHR and L. ROSENFELD: *Det Kgl. Dans. Videnskab. Selskab*, **12**, 8 (1933).

distributions of charge, and not point charges; *i.e.*, not the field components taken at definite space-time points as treated by LANDAU and PEIERLS. Actually quantum electrodynamics is itself independent of the atomic structure of matter, as is apparent from the fact that Planck's constant and the velocity of light are the only constants which are insufficient to fix space-time dimensions appearing in the commutation relations of electromagnetic field strengths. Only by introducing the rest mass of the elementary particle can one fix the space-time dimension. By using extended test bodies whose linear dimensions are chosen large enough compared to atomic dimensions, one can regard their charge density to be approximately constant. Then the radiation reactions of the test body on itself, which would be uncontrollably large if we treated the problem as LANDAU and PEIERLS do, will now be almost negligible. As emphasized by BOHR and ROSENFELD, the passage from classical field theory, which is characterized by the measurability of field strengths at definite space-time points by means of a point charge, to quantum electrodynamics, in which the field strengths are represented by the average values of the idealized field strengths over space-time regions, corresponds to the transition from a point charge to an extended one.

Some progress has been made during the past few years in the theory of quantization of the gravitational field, such as linear approximation by LAURENT [3] and GUPTA [4], Feynman quantization by MISNER and ROSEN [5] and quantization by the canonical formalism by DIRAC [6]. We shall study the question of measurability of gravitational field strengths in the case of the linearized theory, *i.e.*, consider a Lorentz space and treat the actual metric of the Riemannian manifold deviating from this flat case as small. In this theory the commutation relations of the quantized gravitational field strengths are c-numbers containing three universal constants—namely, the universal gravitational constant G, Planck's constant h, and the velocity of light c. Similar to the case in quantum electrodynamics, one characterizes quantum gravitodynamics by the quantized gravitational field strengths, *i.e.*, the quantized Riemannian tensors, which are no longer represented by means of definite space-time point functions but by means of functions averaged over space-time regions. Therefore, we are not concerned with the atomic structure of matter.

It is significant that the usual description of a gravitational field in terms of the Riemann curvature tensors at definite space-time points, which are characteristic of classical general relativity and which are measurable by means of an idealized point particle, cannot directly apply to quantum theory. By

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[3] B. E. LAURENT: *Nuovo Cimento*, **6**, 1445 (1946).

[4] S. N. GUPTA: *Rev. Mod. Phys.*, **29**, 334 (1957) and papers cited there.

[5] C. W. MISNER: *Rev. Mod. Phys.*, **29**, 497 (1957); G. ROSEN: *Ph. D. Thesis* (Princeton, 1959).

[6] P. A. M. DIRAC: *Proc. Roy. Soc. London*, A **246**, 333 (1958).

passing to quantum gravitodynamics in which the quantized gravitational field strengths, or the quantized Riemann curvature tensors, are no longer regarded as finite space-time point functions, but as functions of space-time regions, one finds the way to avoid the difficulties associated with a point mass, such as the infinite self energies of elementary particles inherited from classical field theory.

Hence, the purpose of the present work is to study whether the limitations on the measurability of gravitational field strengths agree with those determined by physically possible measurements so as to give a basis for the validity of quantizing the gravitational field.

In a manner analogous to the electromagnetic formalism, the Riemann tensors can be divided into two classes: The gravitational field strengths of the electric-type components and those of the magnetic-type components. In the measurement of field averages of the magnetic-type components, one has to measure the momentum imparted to a rigid test body. Since the measurement is very similar to that in quantum electrodynamics, the magnetic-type components are to be averaged over a space-time region in the usual sense. However, in the measurement of field averages of the electric-type components, for which one requires a quite different type of physical object—the measurement of the strain of a test body, the field averages over a space-time region are no longer averages in the ordinary sense, but are weighted averages over a space-time region.

If one ignores all limitations inherent in the atomic structure of the test body, then despite two kinds of field averages required in the mesurement of gravitational field strengths, a complete agreement between the limitations on the measurability of field averages and the quantum gravitational formalism can actually be shown.

Similar to quantum electrodynamics, in which the formalism is understood in perturbation expansions in power series of the fine structure constant $e^2/\hbar c$, one has in the quantum gravitational formalism expansions in power series of the constant $Gm^2/\hbar c$, where m denotes the mass of an elementary particle. In the first approximation, therefore, the formalism is understood by quantizing a given pure gravitational field, and not by quantizing the interacting fields between a test body and the given gravitational field. The second approximation, which is not treated in this paper, is the quantization of the field associated with mass particles. The radiative corrections (*), which do not essentially affect the conclusions attained in the first approximation, can be neglected. Their neglect can be justified in this case much better than in the case of quantum electrodynamics because the constant $Gm^2/\hbar c$ is very

(*) The radiative correction in the case of the quantum electromagnetic formalism has been treated by E. CORINALDESI (7).

(7) E. CORINALDESI: *Suppl. Nuovo Cimento*, **10**, 83 (1953).

small: of the order of magnitude 10^{-39} taking the value of the mass of a nucleon, *i.e.*, $m \sim 10^{-24}$ g.

In Section 2 Einstein's theory of general relativity in the linear approximation is reviewed and commutators for the quantized field components as definite space-time point functions are given. In Section 3 some treatment of the quantum fluctuations is carried out and commutators between field averages are explicitly written down. Assumptions required for a test body in order to measure the electric-type components and a detailed analysis of the measuring process are given in Section 4, while a similar treatment for the magnetic-type components is given in Section 5. The uncertainties of field components in one region induced by the measurement of an electric-type component in another region are treated in Section 6 and those induced by the measurement of a magnetic-type component are discussed in Section 7. In Section 8 the measurement of a single field average is given. This is extended to the measurement of two field averages of the same type in Section 9 and of different types in Section 10. Finally, Section 11 is devoted to the study of the measurability in the more general case where three respective principal axes of two cubical test bodies are not parallel to each other. In every case the agreement between the quantum gravitational formalism and the limitations imposed on the measurability of the test body is attained successfully.

2. – Commutators for the quantized gravitational field components in the linearized theory.

Even though several theories of gravitation have been proposed, the most widely accepted is the theory of general relativity by Einstein. One denotes the quadratic form of the line element of the 4-dimensional Riemannian manifold by $\mathrm{d}S^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$, where the Greek indices take the values 1, 2, 3, 4 and the space-time co-ordinates are denoted by $x_\mu = (x_1, x_2, x_3, ict)$, Einstein's classical field equations in the presence of matter are given by

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = -\frac{K}{2}\, T_{\mu\nu}\,, \tag{2.1}$$

with

$$K = \frac{16\pi G}{c^4}\,,$$

where $R_{\mu\nu}$ are the Ricci tensors, $R = R^\mu{}_\mu$ the curvature scaler, $T_{\mu\nu}$ the energy-momentum stress tensors of matter, G the universal gravitational constant, and c the velocity of light. Eq. (2.1) can also be written in the form

$$R_{\mu\nu} = -\frac{K}{2}\,(T_{\mu\nu} - g_{\mu\nu}\, T)\,, \qquad T = T^\mu{}_\mu\,, \tag{2.2}$$

where the usual summation convention is applied to the indices which appear twice.

As one knows that the field equations are non-linear with respect to the field variables $g_{\mu\nu}$, it is possible to linearize the theory with a considerable degree of accuracy. Hereafter we will limit our discussion to the linearized theory unless otherwise stated. In the linearized theory we shall assume that the deviations of the actual metric from that of flat space are so small that the actual metric can be approximated by $g_{\mu\nu} = \delta_{\mu\nu} + \varphi_{\mu\nu}$, where $\delta_{\mu\nu}$ is the Minkowskian metric tensor which has the components

$$\delta = \begin{pmatrix} +1 & & & \\ & +1 & & \\ & & +1 & \\ & & & -1 \end{pmatrix} \tag{2.3}$$

and $\varphi_{\mu\nu}$ are small quantities of the first order. The Riemann tensors are given by

$$R_{\mu\nu\sigma\tau} = \tfrac{1}{2}(\varphi_{\mu\sigma,\nu\tau} - \varphi_{\mu\tau,\nu\sigma} - \varphi_{\nu\sigma,\mu\tau} + \varphi_{\nu\tau,\mu\sigma}) , \tag{2.4}$$

where the commas followed by indices mean ordinary derivatives. It is worthwhile to note that eq. (2.4) is invariant under a gauge transformation $\bar{\varphi}_{\mu\nu} = \varphi_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$ which is induced by co-ordinate transformations of the type $\bar{x}_\mu = x_\mu + \xi_\mu(x_\nu)$ which change the co-ordinate values by small quantities of the first order.

By repeatedly using the Bianchi's identities

$$R_{\mu\nu\sigma\tau,\varrho} + R_{\mu\nu\tau\varrho,\sigma} + R_{\mu\nu\varrho\sigma,\tau} = 0 \tag{2.5}$$

and referring to Einstein's eq. (2.1), one obtains

$$\begin{aligned} \Box^2 R_{\mu\nu\sigma\tau} = -\frac{K}{2}\,[&(T_{\nu\tau} - \tfrac{1}{2}\delta_{\nu\tau}\,T)_{,\mu\sigma} - (T_{\nu\sigma} - \tfrac{1}{2}\delta_{\nu\sigma}\,T)_{,\mu\tau} - \\ &- (T_{\mu\tau} - \tfrac{1}{2}\delta_{\mu\tau}\,T)_{,\nu\sigma} + (T_{\mu\sigma} - \tfrac{1}{2}\delta_{\mu\sigma}\,T)_{,\nu\tau}] , \end{aligned} \tag{2.6}$$

the solution of which is given by

$$\begin{aligned} R_{\mu\nu\sigma\tau}(x) = -\frac{K}{2}\int \mathrm{d}^4x'\,\bar{D}(x - x')[&(T_{\nu\tau} - \tfrac{1}{2}\delta_{\nu\tau}\,T)_{,\mu\sigma} - (T_{\nu\sigma} - \tfrac{1}{2}\delta_{\nu\sigma}\,T)_{,\mu\tau} - \\ &- (T_{\mu\tau} - \tfrac{1}{2}\delta_{\mu\tau}\,T)_{,\nu\sigma} + (T_{\mu\sigma} - \tfrac{1}{2}\delta_{\mu\sigma}\,T)_{,\nu\tau}] , \end{aligned} \tag{2.7}$$

where $\bar{D}(x - x')$, the Green's function of the two space-time points x and x', satisfies the equation

$$\Box^2\,\bar{D}(x - x') = \delta^4(x - x') . \tag{2.8}$$

In the absence of matter, the classical linearized field equations can be written in the following form (*)

$$R_{ij} = 0 , \qquad R_{ijk0} = 0 . \tag{2.9}$$

(*) Latin indices take the values 1, 2, 3.

As a preliminary for quantization, it is more convenient to divide the gravitational field strengths into two parts; namely, the electric-type components E_{ij} and the magnetic-type component H_{ij}, which are defined by

$$E_{ij} = R_{i4j4}\,, \qquad H_{ij} = \frac{i}{2}\varepsilon_{ikl}R_{klj4} = -\frac{\varepsilon_{ikl}}{2}R_{klj0}\,, \tag{2.10}$$

where ε_{ikl} is the 3-dimensional permutation symbol. It is worth-while to mention that the Riemann tensors are symmetric, *i.e.*, $H_{ii}=0$, $E_{ij}=E_{ji}$, and that by eq. (2.9), $E_{ii}=0$ and $H_{ij}=H_{ji}$.

So far we have limited our discussion to classical general relativity theory. By applying quantum theory to general relativity, the quantized Riemann tensors, or the quantized gravitational field strengths, are no longer regarded as c-numbers, but are regarded as operators which operate on a state vector. The quantized Riemann tensors at two space-time points (x, y, z, t) and (x', y', z', t') (*) no longer commute. The commutators have been given by Dr. B. S. DeWitt (8) as

$$\begin{aligned}[R_{\mu\nu\sigma\tau}(x),\, R_{\alpha\beta\gamma\delta}(x')] = \frac{i\hbar}{4}\,Kc\big[&(\delta_{\mu\alpha}\delta_{\sigma\gamma}+\delta_{\mu\gamma}\delta_{\sigma\alpha}-\delta_{\mu\sigma}\delta_{\alpha\gamma})\,D_{,\mu\tau\beta\delta}(x-x') - \\ &-(\delta_{\mu\alpha}\delta_{\tau\gamma}+\delta_{\mu\nu}\delta_{\tau\alpha}-\delta_{\mu\tau}\delta_{\alpha\gamma})\,D_{,\nu\sigma\beta\delta}(x-x') - \\ &-(\delta_{\nu\alpha}\delta_{\sigma\gamma}+\delta_{\nu\gamma}\delta_{\sigma\alpha}-\delta_{\nu\sigma}\delta_{\alpha\gamma})\,D_{,\mu\tau\beta\delta}(x-x') + \\ &+(\delta_{\nu\alpha}\delta_{\tau\gamma}+\delta_{\nu\gamma}\delta_{\tau\alpha}-\delta_{\nu\tau}\delta_{\alpha\gamma})\,D_{,\mu\sigma\beta\delta}(x-x') - \\ &-(\delta_{\mu\beta}\delta_{\sigma\gamma}+\delta_{\mu\gamma}\delta_{\sigma\beta}-\delta_{\mu\sigma}\delta_{\beta\gamma})\,D_{,\nu\tau\alpha\delta}(x-x') + \\ &+(\delta_{\mu\beta}\delta_{\tau\gamma}+\delta_{\mu\gamma}\delta_{\tau\beta}-\delta_{\mu\tau}\delta_{\beta\gamma})\,D_{,\nu\tau\alpha\delta}(x-x') + \\ &+(\delta_{\nu\beta}\delta_{\sigma\gamma}+\delta_{\nu\gamma}\delta_{\sigma\beta}-\delta_{\nu\sigma}\delta_{\beta\gamma})\,D_{,\mu\tau\alpha\delta}(x-x') - \\ &-(\delta_{\nu\beta}\delta_{\tau\gamma}+\delta_{\nu\gamma}\delta_{\tau\beta}-\delta_{\nu\tau}\delta_{\beta\gamma})\,D_{,\mu\sigma\alpha\delta}(x-x') - \\ &-(\delta_{\mu\alpha}\delta_{\sigma\delta}+\delta_{\mu\delta}\delta_{\sigma\alpha}-\delta_{\mu\sigma}\delta_{\alpha\delta})\,D_{,\nu\tau\beta\gamma}(x-x') + \\ &+(\delta_{\mu\alpha}\delta_{\tau\delta}+\delta_{\mu\delta}\delta_{\tau\alpha}-\delta_{\mu\tau}\delta_{\alpha\delta})\,D_{,\nu\sigma\beta\gamma}(x-x') + \\ &+(\delta_{\nu\alpha}\delta_{\sigma\delta}+\delta_{\nu\delta}\delta_{\sigma\alpha}-\delta_{\nu\sigma}\delta_{\alpha\delta})\,D_{,\mu\tau\beta\gamma}(x-x') - \\ &-(\delta_{\nu\alpha}\delta_{\tau\delta}+\delta_{\nu\delta}\delta_{\tau\alpha}-\delta_{\nu\tau}\delta_{\alpha\delta})\,D_{,\mu\sigma\beta\gamma}(x-x') + \\ &+(\delta_{\mu\beta}\delta_{\sigma\delta}+\delta_{\mu\delta}\delta_{\sigma\beta}-\delta_{\mu\sigma}\delta_{\beta\delta})\,D_{,\nu\tau\alpha\gamma}(x-x') - \\ &-(\delta_{\mu\beta}\delta_{\tau\delta}+\delta_{\mu\delta}\delta_{\tau\beta}-\delta_{\mu\tau}\delta_{\beta\delta})\,D_{,\nu\sigma\alpha\gamma}(x-x') - \\ &-(\delta_{\nu\beta}\delta_{\sigma\delta}+\delta_{\nu\delta}\delta_{\sigma\beta}-\delta_{\nu\sigma}\delta_{\beta\delta})\,D_{,\mu\tau\alpha\gamma}(x-x') + \\ &+(\delta_{\nu\beta}\delta_{\tau\delta}+\delta_{\nu\delta}\delta_{\tau\beta}-\delta_{\nu\tau}\delta_{\beta\delta})\,D_{,\mu\sigma\alpha\gamma}(x-x')\big]\,, \end{aligned} \tag{2.11}$$

where $\hbar$ denotes Planck's constant divided by 2π.

(*) We may also denote two space-time points by (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2).

(8) B. S. DeWitt: unpublished manuscript.

Consequently the commutators between the gravitational field of electric-type components and those between the gravitational field of magnetic-type components are given by

$$[E_{ij}(x), E_{kl}(x')] = [H_{ij}(x), H_{kl}(x')] = \tag{2.12}$$
$$= \frac{i\hbar}{4} Kc(d_{ik}d_{jl} + d_{il}d_{jk} - d_{ij}d_{kl})\nabla^4 D(x - x') ,$$

and the commutators between electric-type components and magnetic-type components are

$$[E_{ij}(x), H_{kl}(x')] = -\frac{i\hbar}{4} Kc(d_{ik}\varepsilon_{jlm} + d_{jk}\varepsilon_{ilm} - d_{ij}\varepsilon_{klm}) \frac{\partial^2\nabla^2}{c\,\partial t\,\partial x^m} D(x - x') , \tag{2.13}$$

where

$$d_{ik} = \delta_{ik} - \frac{\partial}{\partial x^i} \nabla^{-2} \frac{\partial}{\partial x^k} , \qquad \nabla^2 = \frac{\partial^2}{\partial x^i\,\partial x^i} . \tag{2.14}$$

In the case of equal times of two field components, these become

$$[E_{ij}(\boldsymbol{\gamma}), E_{kl}(\boldsymbol{\gamma})] = [H_{ij}(\boldsymbol{\gamma}), H_{kl}(\boldsymbol{\gamma}')] = 0 \tag{2.15}$$

and

$$[E_{ij}(\boldsymbol{\gamma}), H_{kl}(\boldsymbol{\gamma}')] = \frac{i\hbar}{4} Kc(-d_{ik}\varepsilon_{jlm} - d_{jk}\varepsilon_{ilm} + d_{ij}\varepsilon_{klm}) \frac{\partial\nabla^2}{\partial x^m} \delta(\boldsymbol{\gamma} - \boldsymbol{\gamma}') . \tag{2.16}$$

Here $E_{ij}(x)$ and $H_{kl}(x)$ denote the gravitational field components of the electric type and the magnetic type respectively at the space-time point (x, y, z, t).

Our approach to field quantization, in which the first approximation is the quantization of pure source-free gravitational fields, enables us to study the measurability of the quantum gravitational formalism from the commutation relations as given above.

The lack of commutability of the field components which can be regarded as the typical character of quantum gravitodynamics is a pure effect of the unity of classical general relativity with quantum field theory. It has no connection with the problems of elementary particles such as singularities due to their self-energies because we can neglect the atomic structure of matter. The use of the test body as a measuring apparatus is always restricted to the extent to which one can handle its behaviour in the given field, as well as its reaction to the field on the basis of classical general relativity.

As we have emphasized earlier, the quantum theoretical field strength has to be considered as a space-time average instead of a point function of the

classical field strength; in the case of gravitational fields it is of decisive importance to consider two kinds of field averages over a space-time region. The first kind is an average of a magnetic-type components $H(x)$ over a space-time region R in the ordinary meaning, denoted by $\tilde{H}^{(R)}$

$$\tilde{H}^{(R)} = \frac{1}{L^3 T} \int_{-L/2}^{L/2} \mathrm{d}x \int_{-L/2}^{L/2} \mathrm{d}y \int_{-L/2}^{L/2} \mathrm{d}z \int_{T} \mathrm{d}t\, H(x)\,, \tag{2.17}$$

and the second kind is a weighted average of an electric-type component $E(x)$ over a space-time region R, denoted by

$$\bar{E}^{(R)} = \frac{6}{L^5 T} \int_{-L/2}^{L/2} \mathrm{d}x \int_{-L/2}^{L/2} \mathrm{d}y \int_{-L/2}^{L/2} \mathrm{d}z \int_{T} \mathrm{d}t \left(x - x_c + \frac{L}{2}\right)\left(\frac{L}{2} - x + x_c\right) E(x)\,, \tag{2.18}$$

where L is the linear dimensions of the cubital space region V, x_c the x coordinate of the center of mass of a test body covering this region, and T the time interval of the space-time region R.

3. – Commutators for field averages.

In the preceding section we have established the commutators of field components, which are regarded as definite space-time functions. As emphasized earlier, we will ignore the atomic structure of the test body and hence we shall need the commutator between field averages. However, there are two kinds of field averages which play an essential role in the measurement of gravitational field strengths; the electric-type components are always taken as weighted averages over a space-time region, while the magnetic components are taken as ordinary averages over a space-time region.

By taking weighted averages for the electric-type components and ordinary averages for the magnetic-type components of the commutators given by (2.12) and (2.13) properly over two given space-time regions I and II according to (2.17) and (2.18), we get the commutation relations between two averages of electric-type components

$$[\bar{E}^{(\mathrm{I})}_{ij}, \bar{E}^{(\mathrm{II})}_{kl}] = \frac{9i\hbar K}{L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^5 T_{\mathrm{II}} c} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x, x') \cdot \tag{3.1}$$

$$\cdot (d_{ik} d_{jl} + d_{il} d_{jk} - d_{ij} d_{kl}) \nabla^4 D(x' - x)\,,$$

and the commutation relations between a weighted average of an electric-type component and an ordinary average of a magnetic-type component

$$(3.2)\qquad [\bar{E}^{(\mathrm{I})}_{ij},\ \tilde{H}^{(\mathrm{II})}_{kl}] = \frac{3i\hbar K}{2L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x) \cdot$$

$$\cdot (-d_{ik}\varepsilon_{jlm} - d_{jk}\varepsilon_{ilm} + d_{ij}\varepsilon_{klm}) \frac{\partial^2}{c\,\partial t\,\partial x^m} \nabla^4 D(x'-x)\,,$$

and finally the commutation relations between two averages of magnetic-type components

$$(3.3)\qquad [\tilde{H}^{(\mathrm{I})}_{ij},\ \tilde{H}^{(\mathrm{II})}_{kl}] = \frac{i\hbar K}{4L_{\mathrm{I}}^3 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x' (d_{ik}d_{jl} + d_{il}d_{jk} - d_{ij}d_{kl})\, \nabla^4 D(x'-x)\,.$$

Here

$$(3.4)\qquad \begin{cases} \omega(x, x') = \left(x - x_c + \dfrac{L_{\mathrm{I}}}{2}\right)\left(\dfrac{L_{\mathrm{I}}}{2} - x + x_c\right)\left(x' - x'_c + \dfrac{L_{\mathrm{II}}}{2}\right)\left(\dfrac{L_{\mathrm{II}}}{2} - x' + x'_c\right), \\ \omega(x) = \left(x - x_c + \dfrac{L_{\mathrm{I}}}{2}\right)\left(\dfrac{L_{\mathrm{I}}}{2} - x + x_c\right), \end{cases}$$

and

$$(3.5)\qquad \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x' = c^2 \int_{V_{\mathrm{I}}} \mathrm{d}v \int_{T_{\mathrm{I}}} \mathrm{d}t \int_{V_{\mathrm{II}}} \mathrm{d}v' \int_{T_{\mathrm{II}}} \mathrm{d}t'\,.$$

Since we shall always choose center-of-mass co-ordinates for each test body, we shall drop hereafter x_c and x'_c, the co-ordinates of the center of mass of each test body with respect to the space fixed co-ordinates, in expressions (3.4).

Since we are interested only in the commutators between the weighted average of the E_{11} component and all field averages over the space-time region II, we write down the expression explicitly:

$$(3.6)\qquad [\bar{E}^{(\mathrm{I})}_{11},\ \bar{E}^{(\mathrm{II})}_{ij}] = i\hbar(\bar{A}^{(\mathrm{I,II})}_{11ij} - \bar{A}^{(\mathrm{II,I})}_{11ij})$$

and

$$(3.7)\qquad \begin{cases} [\bar{E}^{(\mathrm{I})}_{11},\ \tilde{H}^{(\mathrm{II})}_{11}] = 0\,, \\ [\bar{E}^{(\mathrm{I})}_{11},\ \tilde{H}^{(\mathrm{II})}_{ij}] = i\hbar(\bar{\tilde{B}}^{(\mathrm{I,II})}_{11ij} - \bar{\tilde{B}}^{(\mathrm{II,I})}_{11ji})\,, \end{cases}$$

where i and j in (3.7) can not take the value 1 at the same time.

Here for compactness we set

$$
(3.8)\quad
\begin{cases}
\bar{A}^{(\mathrm{I,II})}_{1111} = \dfrac{9K}{L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^5 T_{\mathrm{II}} c} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x,x')(d_{11}\nabla^2)^2 D^{\mathrm{ret}}(x'-x)\,, \\
\bar{A}^{(\mathrm{I,II})}_{1122} = \dfrac{9K}{L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^5 T_{\mathrm{II}} c} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x,x')(2d_{12}^2 - d_{11}d_{22})\nabla^4 D^{\mathrm{ret}}(x'-x)\,, \\
\bar{A}^{(\mathrm{I,II})}_{1133} = \dfrac{9K}{L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^5 T_{\mathrm{II}} c} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x,x')(2d_{13}^2 - d_{11}d_{33})\nabla^4 D^{\mathrm{ret}}(x'-x)\,, \\
\bar{A}^{(\mathrm{I,II})}_{1112} = \dfrac{9K}{L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^5 T_{\mathrm{II}} c} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x,x')(d_{11}d_{12})\nabla^4 D^{\mathrm{ret}}(x'-x)\,, \\
\bar{A}^{(\mathrm{I,II})}_{1113} = \dfrac{9K}{L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^5 T_{\mathrm{II}} c} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x,x')(d_{11}d_{13})\nabla^4 D^{\mathrm{ret}}(x'-x)\,, \\
\bar{A}^{(\mathrm{I,II})}_{1123} = \dfrac{9K}{L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^5 T_{\mathrm{II}} c} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x,x')(2d_{12}d_{13} - d_{11}d_{23})\nabla^4 D^{\mathrm{ret}}(x'-x)\,,
\end{cases}
$$

and

$$
(3.9)\quad
\begin{cases}
\tilde{\bar{B}}^{(\mathrm{I,II})}_{1122} = -\dfrac{3K}{L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x)(d_{12}\nabla^2)\, \dfrac{\partial^2}{c\,\partial t\,\partial z} D^{\mathrm{ret}}(x'-x)\,, \\
\tilde{\bar{B}}^{(\mathrm{I,II})}_{1133} = \dfrac{3K}{L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x)(d_{13}\nabla^2)\, \dfrac{\partial^2}{c\,\partial t\,\partial y} D^{\mathrm{ret}}(x'-x) = -\tilde{\bar{B}}^{(\mathrm{I,II})}_{1122} \\
\tilde{\bar{B}}^{(\mathrm{I,II})}_{1112} = -\dfrac{3K}{2L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x)(d_{11}\nabla^2)\, \dfrac{\partial^2}{c\,\partial t\,\partial z} D^{\mathrm{ret}}(x'-x)\,, \\
\tilde{\bar{B}}^{(\mathrm{I,II})}_{1113} = \dfrac{3K}{2L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x)(d_{11}\nabla^2)\, \dfrac{\partial^2}{c\,\partial t\,\partial y} D^{\mathrm{ret}}(x'-x)\,, \\
\tilde{\bar{B}}^{(\mathrm{I,II})}_{1123} = -\dfrac{3K}{2L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x)(2d_{12}\varepsilon_{13i} - d_{11}\varepsilon_{23i})\, \dfrac{\partial^2\nabla^2}{c\,\partial t\,\partial x_i} D^{\mathrm{ret}}(x'-x),
\end{cases}
$$

from which the remaining expressions $\bar{A}^{(\mathrm{II,I})}$ and $\tilde{\bar{B}}^{(\mathrm{II,I})}$ can be obtained by the appropriate interchange of the suffixes in the retarded Green's function $D^{\mathrm{ret}}(x'-x)$.

On the other hand, the products of the complementary uncertainties of two weighted field averages follow immediately from eq. (3.6) and (3.7):

$$
(3.10)\qquad \Delta\bar{E}^{(\mathrm{I})}_{11}\,\Delta\bar{E}^{(\mathrm{II})}_{ij} \sim \hbar\,|\,\bar{A}^{(\mathrm{I,II})}_{11ij} - \bar{A}^{(\mathrm{II,I})}_{11ij}\,|
$$

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and

$$\Delta\bar{E}_{11}^{(\mathrm{I})}\,\Delta\tilde{H}_{11}^{(\mathrm{II})} = 0\,, \qquad \Delta\bar{E}_{11}^{(\mathrm{I})}\,\Delta\tilde{H}_{ij}^{(\mathrm{II})} \sim \hbar\,|\,\tilde{\bar{B}}_{11ij}^{(\mathrm{I,II})} - \bar{\tilde{B}}_{11ij}^{(\mathrm{II,I})}\,|\,, \tag{3.11}$$

where i and j in (3.11) can not take the value 1 at the same time.

Quite similar to the case of the quantum electromagnetic formalism (except one has instead of electromagnetic field averages the weighted gravitational field averages of the electric type, which in fact do not affect the characteristics obtained in the former case), several important results follow from the above expressions: The quantities $\bar{A}^{(\mathrm{I,II})}$ and $\tilde{\bar{B}}^{(\mathrm{I,II})}$ change in a continuous way under continuous displacements of the boundaries of the regions I and II as long as the dimensions of the region are finite. The differences $\bar{A}^{(\mathrm{I,II})} - \bar{A}^{(\mathrm{II,I})}$ and $\tilde{\bar{B}}^{(\mathrm{I,II})} - \tilde{\bar{B}}^{(\mathrm{I,II})}$ vanish without discontinuity when the two regions gradually coincide.

We turn now to the commutators between the average of the H_{11} component over the space-time region I and all field averages over the space-time region II. From (3.2) and (3.3) we can write down the commutators explicitly as follows:

$$[\tilde{H}_{11}^{(\mathrm{I})}, \tilde{H}_{ij}^{(\mathrm{II})}] = i\hbar(\tilde{C}_{11ij}^{(\mathrm{I,II})} - \tilde{C}_{11ij}^{(\mathrm{II,I})}) \tag{3.12}$$

and

$$[\tilde{H}_{11}^{(\mathrm{I})}, \bar{E}_{11}^{(\mathrm{II})}] = 0\,, \qquad [\tilde{H}_{11}^{(\mathrm{I})}, \bar{E}_{ij}^{(\mathrm{II})}] = i\hbar(\bar{\tilde{D}}_{11ij}^{(\mathrm{I,II})} - \tilde{\bar{D}}_{11ij}^{(\mathrm{II,I})})\,, \tag{3.13}$$

where i and j in (3.13) can not take the value 1 at the same time.

Here for compactness we set

$$\left\{\begin{aligned}
\tilde{C}_{1111}^{(\mathrm{I,II})} &= \frac{K}{4L_{\mathrm{I}}^3 T_{\mathrm{I}} L_{\mathrm{II}}^3 L_{\mathrm{II}} c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'(d_{11}\nabla^2)^2\,D^{\mathrm{ret}}(x'-x)\,,\\
\tilde{C}_{1122}^{(\mathrm{I,II})} &= \frac{K}{4L_{\mathrm{I}}^3 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'(2d_{12}^2 - d_{11}d_{22})\nabla^4 D^{\mathrm{ret}}(x'-x)\,,\\
\tilde{C}_{1133}^{(\mathrm{I,II})} &= \frac{K}{4L_{\mathrm{I}}^3 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'(2d_{13}^2 - d_{11}d_{33})\nabla^4 D^{\mathrm{ret}}(x'-x)\,,\\
\tilde{C}_{1112}^{(\mathrm{I,II})} &= \frac{K}{4L_{\mathrm{I}}^3 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'\,d_{11}d_{22}\nabla^4 D^{\mathrm{ret}}(x'-x)\,,\\
\tilde{C}_{1113}^{(\mathrm{I,II})} &= \frac{K}{4L_{\mathrm{I}}^3 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'\,d_{11}d_{13}\nabla^4 D^{\mathrm{ret}}(x'-x)\,,\\
\tilde{C}_{1123}^{(\mathrm{I,II})} &= \frac{K}{4L_{\mathrm{I}}^3 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'(2d_{12}d_{13} - d_{11}d_{23})\nabla^4 D^{\mathrm{ret}}(x'-x)\,,
\end{aligned}\right. \tag{3.14}$$

and

$$
(3.15)\quad
\begin{cases}
\bar{\tilde{D}}^{(\mathrm{I,II})}_{1122} = \dfrac{3K}{L_{\mathrm{I}}^3 T_{\mathrm{I}} L_{\mathrm{II}}^5 T_{\mathrm{II}} c} \displaystyle\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x')\, d_{12}\, \nabla^2 \frac{\partial^2}{c\,\partial t\,\partial z}\, D^{\mathrm{ret}}(x'-x)\,, \\
\bar{\tilde{D}}^{(\mathrm{I,II})}_{1133} = -\dfrac{3K}{L_{\mathrm{I}}^3 T_{\mathrm{I}} L_{\mathrm{II}}^5 T_{\mathrm{II}} c} \displaystyle\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x')\, d_{13}\, \nabla^2 \frac{\partial^2}{c\,\partial t\,\partial z}\, D^{\mathrm{ret}}(x'-x)\,, \\
\bar{\tilde{D}}^{(\mathrm{I,II})}_{1112} = \dfrac{3K}{2L_{\mathrm{I}}^3 T_{\mathrm{I}} L_{\mathrm{II}}^5 T_{\mathrm{II}} c} \displaystyle\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x')\, d_{11}\, \nabla^2 \frac{\partial^2}{c\,\partial t\,\partial z}\, D^{\mathrm{ret}}(x'-x)\,, \\
\bar{\tilde{D}}^{(\mathrm{I,II})}_{1113} = -\dfrac{3K}{2L_{\mathrm{I}}^3 T_{\mathrm{I}} L_{\mathrm{II}}^5 T_{\mathrm{II}} c} \displaystyle\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x')\, d_{11}\, \nabla^2 \frac{\partial^2}{c\,\partial t\,\partial y}\, D^{\mathrm{ret}}(x'-x)\,, \\
\bar{\tilde{D}}^{(\mathrm{I,II})}_{1113} = \dfrac{3K}{2L_{\mathrm{I}}^3 T_{\mathrm{I}} L_{\mathrm{II}}^5 T_{\mathrm{II}} c} \displaystyle\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x')(2d_{12}\varepsilon_{13i} - d_{11}\varepsilon_{23i}) \frac{\nabla_2 \partial^2}{c\partial t \partial x_i}\, D^{\mathrm{ret}}(x'-x)\,,
\end{cases}
$$

from which the $\tilde{C}^{(\mathrm{II,I})}$'s and $\bar{\tilde{D}}^{(\mathrm{II,I})}$'s can be obtained by appropriately interchanging the primed and unprimed quantities in the retarded Green's function $D^{\mathrm{ret}}(x'-x)$ in the above expressions.

The corresponding products of uncertainties are

$$
(3.16)\qquad \Delta\tilde{H}^{(\mathrm{I})}_{11}\, \Delta\tilde{H}^{(\mathrm{II})}_{ij} \sim \hbar\, |\, \tilde{C}^{(\mathrm{I,II})}_{11ij} - \tilde{C}^{(\mathrm{II,I})}_{11ij}\, |
$$

and

$$
(3.17)\qquad \Delta\tilde{H}^{(\mathrm{I})}_{11}\, \Delta\bar{E}^{(\mathrm{II})}_{11} = 0\,, \qquad \Delta\tilde{H}^{(\mathrm{I})}_{11}\, \Delta\bar{E}^{(\mathrm{II})}_{ij} \sim \hbar\, |\, \bar{\tilde{D}}^{(\mathrm{I,II})}_{11ij} - \bar{\tilde{D}}^{(\mathrm{II\,I})}_{11ij}\, |\,,
$$

where i and j in (3.17) can not take the value 1 at the same time. Just as in the discussion in the preceding part of this section on the $\bar{A}^{(\mathrm{I,II})}$'s and $\bar{\tilde{B}}^{(\mathrm{I,II})}$'s, the $\tilde{C}^{(\mathrm{I,II})}$'s and $\bar{\tilde{D}}^{(\mathrm{I,II})}$'s have the same property of continuity under continuous displacements of the boundaries; also the differences $\tilde{C}^{(\mathrm{I,II})} - \tilde{C}^{(\mathrm{II,I})}$ and $\bar{\tilde{D}}^{(\mathrm{I,II})} - \bar{\tilde{D}}^{(\mathrm{II,I})}$ vanish continuously if the boundaries of the two regions are brought gradually to coincide with each other.

From (3.6) and (3.7) (or (3.12) and (3.13)) it follows that the weighted averages of any two electric-type components (or magnetic-type components) over the same space-time region commute with each other and therefore can always be measured simultaneously with an arbitrary accuracy independent of one another. This can be seen from the symmetry properties of the integrands in relations (3.6) and (3.7) (or (3.12) and (3.13)), say $\bar{A}^{(1,2)} - \bar{A}^{(2,1)}$ (*) and $\bar{\tilde{B}}^{(1,2)} - \bar{\tilde{B}}^{2,1)}$, as follows. If the two time regions are identical, $T_{\mathrm{I}} = T_{\mathrm{II}}$, from

(*) We have denoted any one of the integrands of (3.8) by $\bar{A}^{(1,2)}$ and similarly for $\bar{\tilde{B}}^{(1,2)}$, $\tilde{C}^{(1,2)}$ and $\bar{\tilde{D}}^{(1,2)}$.

the antisymmetric properties of the differences $\bar{A}^{(1,2)}-\bar{A}^{(2,1)}$ (or $\bar{C}^{(1,2)}-\bar{C}^{(2,1)}$) with respect to t_1 and t_2 we find that the time integrals in the differences $\bar{A}^{(\mathrm{I},\mathrm{II})}-\bar{A}^{(\mathrm{II},\mathrm{I})}$ (or $\bar{C}^{(\mathrm{I},\mathrm{II})}-\bar{C}^{(\mathrm{II},\mathrm{I})}$) vanish. Hence the weighted field averages (or the ordinary field averages) of the same type over different spatial regions commute provided their associated time intervals coincide. Therefore

$$\Delta\bar{E}_{ij}^{(\mathrm{I})}(T)\,\Delta\bar{E}_{kl}^{(\mathrm{II})}(T)=0\,,\qquad \Delta\tilde{H}_{ij}^{(\mathrm{I})}(T)\,\Delta\tilde{H}_{kl}^{(\mathrm{II})}(T)=0\,. \tag{3.18}$$

If the two spatial regions are identical, $V_{\mathrm{I}}=V_{\mathrm{II}}$, then owing to the symmetry of the Green's function $\bar{D}(x_1-x_2)$ under interchange of the spatial points (x_1, y_1, z_1) and (x_2, y_2, z_2) together with the antisymmetry of the derivatives in the differences $\bar{B}^{(1,2)}-\bar{B}^{(2,1)}$ (or $\bar{\tilde{D}}^{(1,2)}-\bar{\tilde{D}}^{(1,2)}$) with respect to the spatial points, it may be inferred that the two field averages of different types over the same spatial volume but arbitrary time regions commute. Thus

$$\Delta\bar{E}_{ij}^{(\mathrm{I})}(V)\,\Delta\tilde{H}_{kl}^{(\mathrm{II})}(V)=0\,. \tag{3.19}$$

From (3.18) and (3.19) we get the required result that any two field components averaged over the same space-time region can be accurately measured simultaneously.

However, if one follows the corresponding process treated by HEISENBERG (*), in which a test body is considered to be a point particle in the limiting case, by the same reasoning as in the quantum electromagnetic formalism the differences $\bar{\tilde{B}}^{(\mathrm{I},\mathrm{II})}-\bar{\tilde{B}}^{(\mathrm{II},\mathrm{I})}$ (or $\bar{\tilde{D}}^{(\mathrm{I},\mathrm{II})}-\bar{\tilde{D}}^{(\mathrm{II},\mathrm{I})}$) suffer discontinuous changes under continuous relative displacement of the two regions I and II. Due to this discontinuity the averages of two field components of different types over the same space region do not commute. Since we have ignored the atomic structure of matter we are not interested in this case, but are interested only in the case where the spatial dimensions, the order of magnitude of which we shall denote hereafter by L, is much larger than c multiplied by the time dimension T.

The case $L\leqslant cT$ is actually of little interest in the domain in which classical theory is valid because upon taking a spatial average of the field over a region V the peculiarities of the field cancel out. An analogous cancellation of the fluctuation phenomena occurs in principle in the quantum domain. The statistical fluctuation resulting from the character of the formalism therefore plays a minor role in the case $L>ct$, but enters in essentially for the case $L\leqslant ct$, which we shall discuss below.

The distribution density $F(p)$ of a given graviton with isotopic polarization and given momentum $\hbar\boldsymbol{p}$ and energy $E=\hbar cp$ represented in the form

$$\delta_{AB}F(p)\,\delta(\boldsymbol{p}-\boldsymbol{p}')=\frac{1}{E}\langle\Psi|a_A^{+}(\boldsymbol{p})a_B(\boldsymbol{p}')|\Psi\rangle\,, \tag{3.20}$$

(*) W. HEISENBERG: *Physical Foundation of Quantum Theory* (1930).

in terms of the annihilation and creation operators of a graviton $a_\lambda^-(\boldsymbol{p})$ and $a_\lambda(\boldsymbol{p})$ belonging to a definite polarization λ of momentum state $\boldsymbol{p}$ and $|\Psi$ belonging to the state with definite graviton composition.

Although the expectation value of a weighted field average, for example E_{11}, vanishes, its expectation value of the mean square fluctuation as well as its vacuum expectation value of the mean square fluctuation does not remain finite but diverges.

Since we are not interested in the classical domain, we will not discuss the problem of fluctuations further, but will turn our attention to the truly quantum domain of which the critical field strength u, which is defined as the square root of the order of magnitude of expression (3.10), can be found by a simple order of magnitude estimation in either case; when $L < cT$

$$\sqrt{\Delta\bar{E}_{ij}^{(\mathrm{I})}\Delta\bar{E}_{kl}^{(\mathrm{II})}} \sim \sqrt{\frac{\hbar G}{c^5 L^4 T^2}} = u\,, \tag{3.21}$$

and when $L > cT$,

$$\sqrt{\Delta\bar{E}_{ij}^{(\mathrm{I})}\Delta\bar{E}_{kl}^{(\mathrm{II})}} \sim \sqrt{\frac{\hbar G}{c^4 L^5 T}} = u\,. \tag{3.22}$$

In the case where field strengths are larger than u we reach the domain of the validity of classical general relativity in which quantum features lose their significance.

4. – Measurement of electric-type field averages.

The measurement of electric-type gravitational field strengths appears quite different from that of electromagnetic field strengths. It is better to start our discussion in a manifestly covariant form and later on go to the linearized theory.

It is more convenient to begin by considering two adjacent particles of mass ϱ which describe in space-time world lines C and C'', which are not geodesics, in general, due to their mutual interaction. If we signify C' and C'' respectively by $x(\tau)$ and $x'(\tau')$, τ and τ' being proper times along C and C'' respectively, and set up a one-to-one correlation between points of the two curves and introduce the vector $\eta^\mu = x^{\mu'} - x^\mu$, we get

$$\frac{\delta^2\eta^\mu}{\delta\tau^2} = (u^\mu{}_{,\nu}u^\nu)_{,\sigma}\eta^\sigma + R^\mu{}_{\nu\sigma\tau}u^\nu u^\sigma\eta^\tau\,, \tag{4.1}$$

where $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$, $\delta/\delta\tau$ denotes absolute differentiation and the dots followed by indices, covariant differentiation. Thus far, we have considered the deviation of co-ordinates between two idealized mass points, the masses of which vanish everywhere except in the interior of the two small world tubes C and C'. These tubes are supposed to be concentrated near the world lines so that their dimensions are much smaller than the dimensions of η^μ.

On the other hand, measurements of gravitational fields require an apparatus consisting of an elastic medium made up of a continuous distribution of mass over an extended volume. In order to treat these macroscopic properties of matter, rather than its microscopic behaviour, one may regard eq. (4.1) as an equation which describes the deviation of co-ordinates between the two centers of mass of two volume elements with mass density ϱ which is uniform throughout the whole volume of a test body. World lines are thus supposed to be the world lines of the centers of mass of these volume elements. Thus we regard eq. (4.1) as the dynamical equation of motion of a system of continuous matter. When no volume forces act on the test body the dynamical behaviour of the elastic medium is, in fact, limited by the conditions

$$(4.2) \qquad (\varrho c^2 u^\mu u^\nu + t^{\mu\nu})_{,\nu} = 0\,, \qquad \text{and} \qquad t^{\mu\nu} u_\nu = 0\,,$$

where $t^{\mu\nu}$ [9] is the internal stress tensor of the continuum which is due to the mutual forces exerted between the particles contained in the two adjacent volume elements. Eq. (4.1) together with (4.2) completely determines the dynamical behaviour of the system. In the case $t^{\mu\nu} = 0$, that is, when interactions between the particles contained in any two volume elements are neglected, C and C' are geodesics. The first term on the right-hand side of eq. (4.1) therefore vanishes and the equation reduces to the equation of geodesic deviation [10]. By eliminating the first order covariant derivatives of u^μ from eq. (4.1) and (4.2) we have [8]

$$(4.3) \qquad (\delta^\mu_\nu + c^{-2} u^\mu u_\nu) \frac{\delta^2 \eta^\nu}{\delta\tau^2} = -\Big\{(\delta^\nu_\mu + c^{-2} u^\mu u_\nu)\big[\varrho^{-1}(\delta^\sigma_\nu + u^\nu u_\sigma) t^{\sigma\tau}{}_{.\tau}\big]_{.\varrho}\, \eta^\varrho \cdot$$

$$+ 2c^{-2}\varrho^{-2}(\delta^\mu_\nu + c^{-2} u^\mu u_\nu) t^{\nu\sigma}{}_{.\sigma}(\delta_{\tau\varrho} + c^{-2} u_\tau u_\varrho) t^{\varrho\lambda}{}_{.\lambda}\, \eta^\tau + R^\mu{}_{\delta\varrho\sigma}\, u^\delta u^\varrho \eta^\sigma \Big\}\,.$$

Up till now we have kept the calculation manifestly covariant. We now linearize the above equation so that hereafter we shall not need to distinguish subscripts and superscripts. Furthermore, if we neglect terms containing η higher than quadratic and if we suppose the velocities of the particles are

[9] P. G. Bergmann: *An Introduction to the Theory of Relativity* (London 1947), chap. VIII.

[10] J. L. Synge and A. Schild: *Tensor Calculus* (Toronto, 1949).

much smaller than the velocity of light, we have

$$\frac{\mathrm{d}^2\eta_i}{\mathrm{d}t^2} = -c^2 R_{i0j0}\eta^j - \varrho^{-1}t_{ik,kj}\eta^j . \tag{4.4}$$

Let us now denote the components of Euler co-ordinates (the space-fixed co-ordinates) of the centers of mass of the two volume elements of the test body by $\bar{x}^i$ and x^i and their differences by $\mathrm{d}x^i$. Then the above equation becomes

$$\ddot{x}_i' - \ddot{\bar{x}}_i = (-c^2 R_{i0j0} - \varrho^{-1}t_{ik,kj})(x_j' - x_j) , \tag{4.5}$$

where the dots denote ordinary differentiations with respect to local time. In Lagrangian co-ordinates (the co-ordinates moving with particles) where the co-ordinate components of elongation are denoted by δx_i, we get

$$x_i'(x_k + \delta x_k) - x_i(x_k) = \mathrm{d}x_i + \delta x_{i,j}\mathrm{d}x^j , \tag{4.6}$$

where $\mathrm{d}x_i$ is independent of time. By substituting (4.6) in (4.5) and neglecting higher order infinitesimal quantities on the right-hand side of (4.5), we get the fundamental dynamical equations of motion

$$\delta\ddot{x}_{i,j} = -c^2 R_{i0j0} - \varrho^{-1}t_{ik,kj} , \tag{4.7}$$

from which we shall start our discussion of the measurability of a given gravitational field by considering limitations imposed by the Uncertainty Principle. In the linearized theory we measure the components of the Riemann tensors because they are invariant in the gauge transformation as discussed after eq. (2.4). Equation (4.7) apparently fits our purpose since it contains the electric-type components E_{ij}.

In measuring gravitational field strengths one finds it indispensable, in order to attribute a definite meaning to the field strength, to assume that the mass distribution of a test body is uniform. Considering that a test body has atomic structure and therefore consists of many elementary particles, it is apparent that the assumption of a uniform mass distribution in the test body is already an idealization which consequently imposes limitations on its applicability.

Let us now consider a cubical test body with linear dimensions L of an elastic medium which not only has a uniform mass distribution, but also is constrained, for the sake of simplicity, rigidly in the y and z directions and constrained to stretch uniformly in the x direction. t_{11} is then just the internal stress needed to maintain this constraint. In order to measure an electric-type

gravitational field component, for example E_{11}, averaged over a volume V and a time interval T, one has to locate the test body to cover the spatial region V. For the measurement of the electric-type component it is also of fundamental significance to consider a hyperbolic weighted field average over the space-time region. Since the field equations are invariant under the gauge transformation $\bar{q}_{\mu\nu} = q_{\mu\nu} + \xi_{\mu,\nu} - \xi_{\nu,\mu}$, we may suppose that the center of mass of the test body is always at the same position insofar as the measurement of electric-type components are concerned.

By putting $\delta x = \varepsilon_x$, $\delta y = 0$ and $\delta z = 0$ into eq. (4.7), one has the equation for measuring the non-uniform electric-type component in a single direction x, with its origin at the center of mass of the test body,

$$\ddot{\varepsilon}_x = c^2 E_{11} - \varrho^{-1} t_{1i,i1} . \tag{4.8}$$

From the boundary condition that the internal stress vanishes on the surface of the test body, $t_{1k} = 0$, and from the condition that the internal forces can do no net work, *i.e.*,

$$\int_{-L/2}^{L/2} \mathrm{d}x \int_{-L/2}^{L/2} \mathrm{d}y \int_{-L/2}^{L/2} \mathrm{d}z\, t_{1k,k}\, \delta x = 0 , \tag{4.9}$$

we find

$$\int_{-L/2}^{L/2} \mathrm{d}x \int_{-L/2}^{L/2} \mathrm{d}y \int_{-L/2}^{L/2} \mathrm{d}z\, \omega(x) t_{1k,k1} = 0 . \tag{4.10}$$

Taking the weighted average of equation (4.8) over the volume V, we obtain the dynamical equation of motion

$$\ddot{\varepsilon}_x = c^2 \underline{E}_{11} , \tag{4.11}$$

where we have used (4.10) together with the expression

$$\underline{E}_{11} = \frac{6}{L^5} \int_{-L/2}^{L/2} \mathrm{d}x \int_{-L/2}^{L/2} \mathrm{d}y \int_{-L/2}^{L/2} \mathrm{d}z\, \omega(x)\, E_{11} , \tag{4.12}$$

and the relation

$$\int_{-L/2}^{L/2} \mathrm{d}x \int_{-L/2}^{L/2} \mathrm{d}y \int_{-L/2}^{L/2} \mathrm{d}z\, \omega(x) = \frac{L^5}{6c^2} \ddot{\varepsilon}_x . \tag{4.13}$$

Here ε_x is by definition the strain, or elongation per unit length in the x di-

rection, which is evidently of decisive importance in the measurement of electric-type components.

The Lagrangian of the test body system is then defined by the total work done on the system, *i.e.*, $\int\int\int\int \varrho \delta\ddot{x}_x \, dx \, dy \, dz \, dt$:

$$\mathscr{L} = \frac{M}{24} L^5 \dot{\varepsilon}_x^2 , \tag{4.14}$$

where M is the total mass of the test body and $\dot{\varepsilon}_x$ the time rate of change of the strain at some instant t. Following the usual procedure for partially differentiating the Lagrangian with respect to the time rate of change of the given canonical variable ε_x, we find π_x, the momentum canonically conjugate to ε_x, as

$$\pi_x = \frac{M}{12} L^2 \dot{\varepsilon}_x . \tag{4.15}$$

By locating the test body to fill the volume V and by measuring the canonically conjugated momenta π_x' and π_x'' of the test body in the x-direction at the beginning t' and the end t'' of the time interval T, we obtain the equation for the balance of the strain momentum of the test body with the weighted field average $\bar{E}_{11}$ over the space-time region

$$\pi_x'' - \pi_x' = \tfrac{1}{12} M L^2 c^2 \bar{E}_{11} T . \tag{4.16}$$

Here it is assumed, of course, that the time intervals required for the strain momentum measurements, whose order of magnitude we shall denote by Δt, can be regarded vanisingly small compared with T; and that the magnitude of ε_x is much smaller than unity. In other words, compared to the linear dimensions L of the volume we can neglect the total elongation δL of the test body system. This elongation results not only from the measurement of the strain momentum, but also from the corresponding acceleration of ε_x imparted to it by the gravitational field, which is to be measured during the time interval T.

Because the strain ε_x and its canonically conjugate momentum π_x of the test body system satisfied the quantum commutation relation

$$[\varepsilon_x, \pi_x] = i\hbar , \tag{4.17}$$

each measurement of the strain momentum π_x carried out with an accuracy $\Delta\pi_x$ is accompanied by a sacrifice in knowledge of its strain to a certain extent

$\Delta\varepsilon_x$ which, according to the Uncertainty Principle, is given by

$$\Delta\varepsilon_x \Delta\pi_x \sim \hbar . \tag{4.18}$$

By using (4.16) and (4.18) and by neglecting Δt and δL compared to T and L respectively, we get the corresponding inaccuracy of the field measurement $\Delta\bar{E}_{11}$ in the order of magnitude

$$\Delta\bar{E}_{11} \sim \frac{12\hbar}{ML^2c^2T\Delta\varepsilon_x} . \tag{4.19}$$

For any value of $\Delta\varepsilon_x$, no matter how small, $\Delta\bar{E}_{11}$ can be measured with arbitrary accuracy if the mass M of the test body is chosen sufficiently large. Aside from the mass of the test body, the accuracy of the field measurement also depends on the absolute magnitude of the value of $\bar{E}_{11}$ itself. This is because for given fringes of Δt and $\Delta\varepsilon_x$ the value of $\bar{E}_{11}$ from (4.16) is affected by an uncertainty of the strain and an uncertainty due to the inaccuracy of the boundaries of the measuring region, both of which go beyond all limit as $\bar{E}_{11}$ becomes infinite. Another difficulty arises when M is large; the accurately measured gravitational field so far described not only contains the external field that we wish to measure, but also the so-called self-reaction of the test body, *i.e.*, the gravitational field produced by the test body itself during the measurement-time interval. The self-reaction is very large if M is very large. We shall discuss this problem later on.

Now let us consider the validity of the linearized theory and hence vacuum fluctuations in a space-time region of dimensions L. Consider the Feynman sum,

$$\sum \exp\left| \frac{ic^3}{\hbar G} \int \left(\frac{\partial g}{\partial x}\right)^2 d^4x \right| , \tag{4.20}$$

where the summation is carried out over all possible histories, the order of magnitude of the exponential only being given. We see that the contributions will add in phase for metric fluctuations smaller than Δg which is given by putting the quantity in the exponential equal to unity:

$$\Delta g^2 \sim \frac{\hbar G}{c^3 L^2} . \tag{4.21}$$

Corresponding fluctuations in the Riemann tensors are

$$\Delta R \sim \sqrt{\frac{\hbar G}{c^3}} \frac{1}{L^3} . \tag{4.22}$$

Thus, the minimum linear dimensions allowed for the validity of the linearized

theory are given by

$$\frac{\hbar G}{c^3 L_{\min}^2} \sim 1 , \tag{4.23}$$

$L_{\min}$ being of the order of 10^{-33} cm.

On the other hand, this is also the limit for which the magnitude of ε_x is much smaller than unity throughout the measuring time interval T, because

$$\Delta R L^2 \sim \frac{L_{\min}}{L} , \tag{4.24}$$

is much smaller than unity if L is much larger than $L_{\min}$. In the domain where L is smaller than $L_{\min}$, there will be a non-linear effect which will go beyond our discussion. As mentioned in Section 3, we are interested only in the truly quantum domain in which we must have

$$\Delta \bar{E}_{ij} \ll u \tag{4.25}$$

for $L > cT$ where $u = L_{\min}/\sqrt{L^5 c \bar{T}}$. Moreover, defining a dimensionless factor λ_e by $\Delta \bar{E}_{ij} \sim \lambda_e u$ we have

$$\lambda_e \sim \frac{12 c^2 L^3}{MG \Delta \varepsilon_x} u , \tag{4.26}$$

which is an important factor for the estimation of the accuracy of measurement. The requirement that λ_e as well as $\Delta \varepsilon_x$ be very small as compared to unity means that the test body must consist of a very large number of elementary particles of mass m, this number being

$$N \sim \frac{M}{m} \sim \frac{12}{\lambda_e \Delta \varepsilon} \sqrt{\frac{L}{cT}} \sqrt{\frac{\hbar c}{m^2 G}} , \tag{4.27}$$

which is very large if $L > cT$ and $\lambda_e \ll 1$ are assumed.

The factor $Gm^2/\hbar c$ whose smallness is of the order of magnitude 10^{-39} taking the value of the mass of a nucleon, *i.e.*, $m \simeq 10^{-24}$ g, appearing in the last expression is the important constant appearing in the quantum gravitational formalism corresponding to the fine structure constant in quantum electrodynamics. This smallness justifies our formalism very successfully. The assumed largeness of N is a necessary condition for the physical realization of a uniform mass distribution of the test body over the space region, and the larger the cubical dimensions L of the test body compared with atomic dimensions, the better it fulfills the physical requirements in principle.

In what follows, we shall discuss the reaction of the test body on itself

during the course of the measurement. Our main problem is to find the most physically attainable measuring arrangement so that the reaction of the test body can be minimized or compensated for to the largest extent. As we have mentioned in Section 1, we are able to treat this reaction classically and not as a quantized field owing to the smallness of $Gm^2/\hbar c$ and as far as it can be classically computed, the actual compensation of the reaction by properly chosen spring mechanisms and their force constants can really be shown.

Since we use an extended test body whose dimensions are much larger than atomic dimensions, the corresponding difficulties on the measurement of the strain momentum, which arise in the electromagnetic formalism as considered by LANDAU and PEIERLS, do not arise in the measurement of either electric-type or magnetic-type components.

Furthermore, let us consider a system of two test bodies. In this case not only a discussion of the self-reaction on each test body, but also a discussion of their mutual influence is necessary. The measurement of a space average of the field reaction, E_{1react}, at a point P_2 of the test body II in volume V_{II} as a result of the strain of the first test body contained in a small volume element dv_1 of the volume V_{I} at a point P_1 is exactly equal and opposite to the measurement of E_{1react}, the field strength produced by the reverse process at P_2. This fact implies an essential feature of the formalism and hence when the boundaries of the two volumes coincide, the mutual influence of the field reaction completely vanishes. Actually in the above discussion of the mutual influence of the disturbance, one may take into account this effect between two small volume elements of one test body. When the strain increases, there will be an increase in the metric which in turn increases an amount of the momentum-energy tensor.

Owing to the fact that in the linearized theory the action is propagated with the finite velocity of light, we have to take into account this effect on the strain so that we are not permitted to regard the uniform stretching of the test body literally, but must regard the test body as a system of constituent particles of small dimensions. We must then think that the measurement of the total strain momentum π_ν of the test body is being carried in such a way that, to a sufficient approximation, each constituent particle suffers a displacement proportional to the magnitude of the x co-ordinate of its location from the center of mass during the strain momentum measurement. Therefore the strain, or elongation per unit length, remains the same on the average. The above requirement can actually be fulfilled so far as we can neglect the atomic structure of the test body. This, in fact, suggests that the strain momentum can be measured on a classical basis, for example, by a study of the Doppler effect in the reflection of photons, provided a bundle of photons sent to the test body is large enough to determine accurately the interaction between the test body and the radiation.

Since the space-time weighted average of the E_{11} component can be written as $12\sum x_\tau(p''_\tau - p'_\tau)/ML^2c^2T$, where the summation is extended over all component particles τ, we know that the strain momentum $\pi_x = \sum x_\tau p_\tau - \sum x_\tau^2 m_\tau \dot{\varepsilon}_x$, where m_τ is the mass of a component particle.

Making use of (4.14) the Hamiltonian of the test body system is given by

$$\mathscr{H} = \frac{6}{ML^2c^2}\pi_x^2 , \tag{4.28}$$

from which it follows that

$$\Delta E = \dot{\varepsilon}_x \Delta\pi_x . \tag{4.29}$$

From this relation together with the uncertainty relation between the time interval Δt and the uncertainty ΔE in the knowledge of the energy, exchanged between the test body and the bundle of photons during the collision process in the measurement, *i.e.*,

$$\Delta E\,\Delta t \sim \hbar , \tag{4.30}$$

we get

$$\Delta\pi_x |\dot{\varepsilon}''_x - \dot{\varepsilon}'_x| \Delta t \sim \hbar . \tag{4.31}$$

The accuracy of the strain may be known within an uncertainty

$$\Delta\varepsilon_x = |\dot{\varepsilon}''_x - \dot{\varepsilon}'_x| \Delta t , \tag{4.32}$$

which is in complete agreement with the uncertainty relation (4.17), regardless of whether $|\dot{\varepsilon}''_x - \dot{\varepsilon}'_x|$ can be known accurately for a sufficiently heavy test body. Recalling ε_x to be the elongation per unit length, we have

$$\Delta\varepsilon_x = \Delta\left(\frac{\delta L}{L}\right), \tag{4.33}$$

which together with the relation $|\varepsilon_x| < 1$ gives rise to the relation

$$\Delta\varepsilon_x \ll \frac{c}{L}\Delta t , \tag{4.34}$$

which limits the absolute accuracy to $\Delta\pi_x$ for a given upper limit Δt in a measurement of the strain momentum.

We can neglect the special-relativistic effect between any pair of adjacent constituent particles of the test body by using sufficiently heavy test bodies so that the relativ e velocities of any pair of adjacent constituent particles are

always much smaller than the velocity of light. Therefore we are also able to regard the uncertainty of the total elongation $L\varepsilon_x$ of the test body, resulting from the strain momentum measurement, as being very small compared with $c\Delta t$, which aslo has to be chosen very small. Consequently, we can arrange the measurement so that

$$|\Delta\varepsilon_x| \ll 1 \tag{4.35}$$

always holds. A test body must be thought to consist of a large number of component particles which interact with a bundle of photons.

In order to measure the strain we use the Doppler effect by optical means quite similar to the considerations of Bohr and Rosenfeld. We imagine each component particle to be equipped with a small mirror which is perpendicular to the x direction. In addition to the mirrors attached to each component particle, we also have properly situated mirrors to make the light path from the light source to each component particle the same so that by sending a bundle of photons to each component particle, we may know the momentum of each component particle, except for an unavoidable uncertainty, through the spectral analysis of the reflected photons. We can now imagine the whole apparatus to be so designed that the number of reflected photons from the mirror of a component particle is proportional to the magnitude of the x co-ordinate of the component particle referred to the center of mass, which is the origin of the co-ordinates of the test body. The other half of the test body is imagined to have equipment similar to the half described above together with the same light source.

We send a bundle of photons of duration Δt so that each component particle will receive a momentum which is proportional to the magnitude of the x co-ordinate from the center of mass of each test body. At the same time different component particles suffer different accelerations. Attention must be paid to the fact that when we send a bundle of photons of a definite frequency from the light source, the frequency will change when it collides with a component particle of the test body because of the gravitational red shift. This correction must be taken into account for a detailed analysis of the measurement, but we will not be concerned with this effect in the following analysis and will simply assume the condition that the stress uncertainty of the radiation bundle is small as compared to that of the test body, *i.e.*, given by the condition $\sqrt{\hbar G/c^3} \ll 1$, which is also the condition for the validity of the linearized theory.

Let us use the same notation as reference [2] and follow the same method. We get

$$v''_{\tau,x} - v'_{\tau,x} = \frac{\hbar}{m_\tau c} \sum_{n_{\tau,x}} (\nu' + \nu'') = \frac{2n_{\tau,x}h\nu_0}{m_\tau c}, \tag{4.36}$$

with

$$v''_{\tau,x+\delta x} - v'_{\tau,x+\delta x} = \frac{2n_{\tau,x+\delta x}\hbar\nu_0}{m_\tau c}, \tag{4.37}$$

where $v'_{\tau,x}$ and $v''_{\tau,x}$ denote velocities before and after collision of a component particle at x, m_τ is its mass and the summation extends over the n_τ photons reflected by the component particle. We have denoted by ν' and ν'' the frequencies of the photons before and after collision with the component particle. The mean frequency ν_0 of the bundle is assumed very large compared to $(\Delta t)^{-1}$ and $\nu' - \nu''$.

Subtracting eq. (4.36) and (4.37) and dividing by the stationary length dx between the two adjacent component particles, we obtain

$$\frac{(v''_{\tau,x+\delta x} - v''_{\tau,x}) - (v'_{\tau,x+\delta x} - v'_{\tau,x})}{dx} = \frac{2 \text{ const } \hbar\nu_0}{m_\tau c}. \tag{4.38}$$

from which by summing over all component particles and dividing by the total stationary length we have

$$\dot{\varepsilon}''_{xx} - \dot{\varepsilon}'_{xx} \quad \dot{\varepsilon}''_{xx}(x) - \dot{\varepsilon}'_{xx}(x) \tag{4.39}$$

which means that the time rate of change of the strain is on the average the same between any two adjacent component particles. The impact of the photons therefore produces an uncertainty of the strain $\Delta\varepsilon_{xx}$, which is uncontrollable but the same between any two adjacent component particles on the average.

By virtue of (4.30), (4.31) and (4.39), we have for the energy of the component particle τ

$$E_\tau = \hbar\Big[\sum_{n_{\tau,x+\delta x}}(\nu' - \nu'') - \sum_{n_{\tau,x}}(\nu' - \nu'')\Big] \tag{4.40}$$

and hence

$$\Delta\pi_{xx}\Delta\varepsilon_{xx} \sim \Delta t\Big\{\sum_\tau \hbar\Big[\sum_{n_{\tau,x+\delta x}}(\nu' - \nu'') - \sum_{n_{\tau,x}}(\nu' - \nu'')\Big]\Big\}. \tag{4.41}$$

Because the summation $\sum_\tau$ extends over all component particles and the difference of the summation, $\sum_{n_{\tau,x+\delta x}} - \sum_{n_{\tau,x}}$, over all adjacent pairs of particles, the part of relation (4.41) inside the curly brackets becomes (*)

$$\sum_{n_{\tau,L/2}} \hbar(\nu' - \nu'') - \sum_{n_{\tau,0}} \hbar(\nu' - \nu'') \tag{4.42}$$

(*) These cancellations are valid only if one can ignore the retardation of forces.

in which the last summation can be ignored, since we can always make an arrangement with $n_{\tau,0}=0$, *i.e.*, no photons being sent to a component particle located at the center of mass of the test body. Then we can write

$$\Delta\pi_x\,\Delta\varepsilon_x \sim \Delta t\,\Delta\Big[\sum_{\tau}\sum_{\nu''_{\tau,L/2}}\hbar(\nu'-\nu'')\Big]\,, \tag{4.43}$$

which is identical with the uncertainty relations (4.17), where $\Delta\varepsilon_x$ is assumed as before to be vanishingly small and uniform over the whole test body.

Despite the reflected radiation being measurable exactly, the incident beam can only be known to an accuracy complementary to Δt. Even though, in general, we cannot obtain information on the strain momentum of any pair of adjacent component particles, it is possible through the spectral analysis of the reflected photons to measure a statistical distribution of the strain momenta which will show a hyperbolic distribution. This fact is closely related to the hyperbolic weight we have to take in the averaging of electric-type components. We must further emphasize that the test body system suffers a uniform stretching under the strain momentum measurement and that this fact is of importance not only in insuring the cancellation of the self reaction, but also in enabling us to regard all test bodies as not suffering strains outside the small time intervals for the strain momentum measurement.

Because the change in strain velocities transferred to each pair of component particles by the initial impact is known, one can compensate for this effect by giving the test body system another oppositely directed impulse. This can be done after the initial strain momentum measurement, still during Δt, without a sacrifice of knowledge concerning the strain momentum of the test body; so that within an uncertainty $L\Delta\varepsilon_x$ the test body covers the space region, the field strength of which we want to measure. After the time interval T following the final measurement, the same process has to be carried out again. During these collision processes it is impossible to know the time interval and the strain more accurately than Δt and $\Delta\varepsilon_x$ respectively.

It is always necessary to know the exact position of every test body before and after each measurement. In order to know exactly the position of each test body before and after its use as a measuring apparatus, outside the measuring time interval T, one binds each component particle of the test body to each component particle of a rigid framework body which occupies nearly the same space region and on which the detecting apparatus of radiation bundles may be mounted. The mass of the framework body is supposed to be much larger than that of the test body so that the oscillation period of the framework body is much longer than the measuring time interval T. The mass of the framework body, however, cannot be too large, otherwise it would violate the condition of the linearized theory, *i.e.*, $M_F G/Lc^2 \ll 1$, where the mass of the framework body is M_F. At the beginning of the time interval

T the binding has to be loosened so that the sum of the strain momentum of each component particle can be measured, while, by an immediate counter impulse, one must be able to bring the test body to rest with a degree of accuracy $L\,1\varepsilon$, allowed by the uncertainty relation. At the end of T, the rigid binding is imposed again on the test body so that it occupies exactly the same original position.

In the measurement of electric-type field averages over two partially overlapping space-time regions, one requires idealized test bodies which can be relatively displaced through each other without mechanical interaction. We are able at least in principle to imagine that the component particles of the test bodies are bound to the corresponding constituent particles of the framework bodies as in the preceding discussion. Only in so far as we can neglect the atomic structure of the test body can we justify these idealized assumptions, as well as compensating spring mechanisms to be introduced later on.

The above discussion has shown that it is possible to distinguish the limitations imposed by the manipulation of the test body, which depend on Planck's constant, from the limitations on the structure of the test body due to the atomic structure of matter. In spite of the fact that the constants appearing in our formalism are sufficient to fix the space-time dimensions, it is required for the validity of the linearized theory that one take the average values of the field strength over a space-time region.

The weight one has to take in the case of electric-type field averages is closely related to the fact that in the radiation of gravitational waves one can always eliminate the dipole radiation by suitably choosing co-ordinates, leaving the quadrupole radiation as the first radiative term.

5. – Measurement of magnetic-type field averages.

The measurement of the magnetic-type gravitational field strength is quite analogous to that of the electric field strength. Consider a spinning test particle of PAPAPETROU [11] whose equations of motion are derived under assumptions on the form of the energy-momentum tensor in a small world tube. In the limit of the tube becoming a line, these equations of motion are given by

$$\frac{\delta}{\delta\tau}\left(mu^{\mu}+u_{\nu}\frac{\delta}{\delta\tau}S^{\mu\nu}\right)+\frac{1}{2}R^{\mu}{}_{\sigma\varrho\tau}u^{\sigma}S^{\varrho\tau}=0\,, \tag{5.1}$$

$$\frac{\delta}{\delta\tau}S^{\varrho\sigma}+u^{\varrho}u_{\tau}\frac{\delta S^{\sigma\tau}}{\delta\tau}-u^{\sigma}u_{\nu}\frac{\delta}{\delta\tau}S^{\varrho\nu}=0\,, \tag{5.2}$$

[11] A. PAPAPETROU: *Proc. Roy. Soc. London*, A **209**, 248 (1951); A. PAPAPETROU and E. CORINALDESI: *Proc. Roy. Soc. London*, A **209**, 259 (1951).

where $S^{\mu\nu}$ is a skew-symmetric tensor expressing the internal angular momentum of the particle. If in addition to these equations we choose the condition ([12]) $S^{\mu\nu}u_\nu = 0$, which is a generalization of the conservation of momentum, we can write (5.1) and (5.2) in the form

$$m\frac{\delta u^\mu}{\delta\tau} - S^{\mu\varrho}\frac{\delta^2 u_\varrho}{\delta\tau^2} + R^\mu{}_{\nu\varrho\sigma}u^\nu S^{\varrho\sigma} = 0\,, \tag{5.3}$$

and

$$\frac{\mathrm{d}m}{\mathrm{d}\tau} = 0\,, \tag{5.4}$$

which expresses the conservation of mass. From eq. (5.3), which is the fundamental equation for a spinning test particle, we shall start our discussion on the measurement of magnetic-type components. If the velocity of the spinning test particle is much smaller than the velocity of light and if we consider the case in which $\delta^2 u_\varrho/\delta\tau^2 = 0$, by linearizing the theory we have

$$m\ddot{x}_i = -\frac{c}{2}R_{i0jk}S_{jk}\,. \tag{5.5}$$

In order to measure the magnetic-type components we have to consider a test body consisting of many spinning particles. The cubical dimensions of the test body must be chosen much larger than the atomic dimensions so that we can assume that the test body has a uniform mass density m. Moreover, we assume that the test body has a spin density with the components $\sigma_x = S_{23} \neq 0$, $\sigma_y = 0$ and $\sigma_z = 0$ distributed over the whole volume so that inside the test body mass currents cancel out one another leaving mass currents only on its surfaces. Then we may write eq. (5.5) as

$$m\ddot{x}_i = -cR_{i023}\sigma_x\,. \tag{5.6}$$

For the measurement of a magnetic-type field strength, for example H_{11}, we need only the equation

$$m\ddot{x} = cH_{11}\sigma_x\,, \tag{5.7}$$

which is exactly the same type of equation as that describing the motion of a charged particle of mass m and charge e in an electric field of strength E_x, if we replace the magnetic-type component H_{11} multiplied by the spin density σ_x by E_x, and replace the velocity of light by the charge e. Taking the space-time

([12]) F. PIRANI: *Acta Phys. Polon.*, **15**, 389 (1956).

average of eq. (5.7), one has

$$p_x'' - p_x' = c\tilde{H}_{11}\Sigma_x T \,. \tag{5.8}$$

We can proceed with the whole discussion on the measurement of the magnetic-type gravitational field in a manner parallel to that of the electric field average according to reference ([2]), but we will drop the topic at this point.

6. – Electric-type field reactions of the test body.

In what follows we shall study the field reactions of the test body which accompany the measurement of the gravitational field strengths and show in fact that these reactions, which are proportional to the strain, can be automatically compensated for by adjusting the spring constants appropriately, just as BOHR and ROSENFELD did in matching the reaction of the electromagnetic field by a mechanical spring. We intend to study the reactions of the test body produced only by a single field measurement in this section. To follow the foregoing procedure we shall assume that every test body has a uniform mass distribution and suffers a uniform stretching during the measurement of the strain momentum.

Let us now consider two space-time regions I and II having volumes V_{I} and V_{II} and time intervals T_{I} and T_{II} so located that light signals emitted from some points of I can reach II but signals from II cannot reach I. We imagine that the volume V_{I} is covered by the first test body and V_{II} by the second test body. The measurement of the weighted averages of E_{11} over the region I produces the gravitational field at a point (x_2, y_2, z_2, t_2) of the region II, by whose effect we can not measure the field strength in the region II accurately. To calculate thisi nduced gravitational fiel dat the point (x_2,y_2,z_2,t_2), we may be able to follow a process corresponding to the calculation done by BOHR and ROSENFELD by finding the metric induced by the energy-momentum stress tensor and then the induced gravitational field, etc., but instead we shall follow a direct method to compute the induced gravitational field by using the solution (2.7).

It is possible to assume that the center of mass of each test body is always at rest in so far as the measurement of electric-type components is concerned so that we can use the center of mass co-ordinates of each test body. At the beginning of T_{I}, in the time interval from t_{I}' to $t_{\text{I}}'+\Delta t_{\text{I}}$, the first test body suffers an increment of the strain $\Delta\varepsilon_x$; it remains at a constant strain $\Delta\varepsilon_x$ during the time interval T_{I}, *i.e.*, $t_{\text{I}}''-t_{\text{I}}'$. At the end of T_{I}, in the time interval from t_{I}'' to $t_{\text{I}}''+\Delta t_{\text{I}}$, it recovers its original shape. We shall assume that Δt_{I} is much smaller than $T_{\text{I}} = t_{\text{I}}''-t_{\text{I}}'$ and that $\Delta\varepsilon_x$ is much smaller than unity, as well as smaller than $c\,\Delta t_{\text{I}}/L$.

Consider now the energy-momentum tensor of the first test body system

accompanied by the strain ε_x, instead of the strain $\Delta\varepsilon_x$, which is generated by a uniform stretching of the test body in the x direction and maintaining it unchanged in the y and z directions. If we denote by L the linear dimensions of the cubical volume, then according to the mass conservation law it follows that the increase of the energy density T_{00} is $-\varrho c^2\varepsilon_x$ throughout the volume V_{I} and is $\varrho c^2 L\varepsilon_x\delta(x-(L/2))/2$ and $\varrho c^2 L\varepsilon_x\delta(x+(L/2))/2$ on the surfaces $x=L/2$ and $x=-L/2$ respectively during the time interval T_{I}. Summing up these quantities, we have for the total energy density

$$T_{00}=\varrho c^2\varepsilon_x\left|-\theta\left(x+\frac{L}{2}\right)\theta\left(\frac{L}{2}-x\right)+\frac{L}{2}\,\delta\left(x-\frac{L}{2}\right)+\frac{L}{2}\,\delta\left(x+\frac{L}{2}\right)\right|\cdot \tag{6.1}$$

$$\cdot\theta(t-t_{\mathrm{I}}')\theta(t_{\mathrm{I}}''-t)\,\theta\left(y+\frac{L}{2}\right)\theta\left(\frac{L}{2}-y\right)\theta\left(z+\frac{L}{2}\right)\theta\left(\frac{L}{2}-z\right),$$

which can be written in the form

$$T_{00}=\frac{\partial^2 P'}{\partial x^2}\,, \tag{6.2}$$

with

$$P'=\frac{\varrho c^2}{2}\varepsilon_x\left(x+\frac{L}{2}\right)\left(\frac{L}{2}-x\right)\theta\left(x+\frac{L}{2}\right)\theta\left(\frac{L}{2}-x\right)\theta\left(y+\frac{L}{2}\right)\theta\left(\frac{L}{2}-y\right)\cdot$$

$$\cdot\theta\left(z+\frac{L}{2}\right)\theta\left(\frac{L}{2}-z\right)\theta\,(t-t_{\mathrm{I}}')\theta(t_{\mathrm{I}}''-t)\,,$$

where ε_x is assumed to be a constant and $\theta(x)$ is a step function with the property that $\theta(x)=1$ if $x<0$ and is zero if $x>0$. Since the momentum density T_{10} is proportional to the magnitude of the x co-ordinate and is generated only at $t=t_{\mathrm{I}}'$ in the x direction and at $t=t_{\mathrm{I}}''$ in the opposite direction, we may write

$$T_{10}=-\varrho c x\varepsilon_x\left[\delta(t-t_{\mathrm{I}}')-\delta(t_{\mathrm{I}}''-t)\right]\theta\left(x+\frac{L}{2}\right)\theta\left(\frac{L}{2}-x\right)\cdot \tag{6.3}$$

$$\cdot\theta\left(y-\frac{L}{2}\right)\theta\left(\frac{L}{2}-y\right)\theta\left(z+\frac{L}{2}\right)\theta\left(\frac{L}{2}-z\right),$$

which also can be written as

$$T_{10}=\frac{\partial^2 P'}{\partial x\,\partial x_0}\,. \tag{6.4}$$

The momentum flux density T_{11}, however, cannot be obtained from physical considerations directly, since we do not know the internal stress tensor t_{ij} of the test body. Nevertheless, we can derive it from the conservation law $\partial T_{\mu\nu}/\partial x_\nu=0$, because all other components are zero. Hence

$$T_{11}=\frac{\partial^2 P'}{\partial x_0^2}\,. \tag{6.5}$$

The energy-momentum tensor of the first test body induced by the uncertainty of its strain $\Delta\varepsilon_x$ is thus given by

$$(6.6)\qquad \Delta T_{00}=\frac{\partial^2 P_e}{\partial x^2},\qquad \Delta T_{10}=\frac{\partial^2 P_e}{\partial x\,\partial x_0},\qquad \Delta T_{11}=\frac{\partial^2 P_e}{\partial x_0^2},$$

with the generalized polarization of the electric type

$$P_e=\frac{\varrho c^2}{2}\Delta\varepsilon_x\omega(x)\,\theta\Big(x+\frac{L}{2}\Big)\theta\Big(\frac{L}{2}-x\Big)\theta\Big(y-\frac{L}{2}\Big)\theta\Big(\frac{L}{2}-y\Big)\cdot$$

$$\cdot\theta\Big(z-\frac{L}{2}\Big)\theta\Big(\frac{L}{2}-z\Big)\theta(t-t_1')\theta(t_1''-t)\,.$$

Replacing the Green's function $\bar{D}(x_2-x_1)$ by the retarded Green's function $D^{\mathrm{ret}}(x_2-x_1)$, the uncertainties of field components at (x_2, y_2, z_2, t_2) due to the increment $\Delta\varepsilon_x$ or the first test body in the region I is

$$(6.7)\quad \Delta R_{\mu\nu\sigma\tau}(x_2)=-\frac{K}{2}\int \mathrm{d}^4x_1 D^{\mathrm{ret}}(x_2-x_1)\big[(\Delta T_{\nu\tau}-\tfrac{1}{2}\delta_{\nu\tau}\Delta T)_{,\mu\sigma}-(\Delta T_{\nu\sigma}-\tfrac{1}{2}\delta_{\nu\sigma}\Delta T)_{,\mu\tau}-$$

$$-(\Delta T_{\mu\tau}-\tfrac{1}{2}\delta_{\mu\tau}\Delta T)_{,\nu\sigma}+(\Delta T_{\mu\sigma}-\tfrac{1}{2}\delta_{\mu\sigma}\Delta T)_{,\nu\tau}\big]\,.$$

For the E_{11} component we have by means of (2.10) and (6.6)

$$(6.8)\quad \Delta E_{11}(x_2)=\frac{K}{8}\int \mathrm{d}^4x_1 D^{\mathrm{ret}}(x_2-x_1)\big[P_{e,0011}+P_{e,1111}+P_{e,0000}+P_{e,1100}-4P_{e,1010}\big]\,.$$

Integrating by parts on the right-hand side of eq. (6.8) (*) and neglecting all terms except those which have fourth order differentiation of the function $D^{\mathrm{ret}}(x_2-x_1)$, we have

$$(6.9)\qquad \Delta E_{11}(x_2)=\frac{K\varrho}{8}\Delta\varepsilon_x\int \mathrm{d}^4x_1\omega(x_1, x_2)\left(\frac{\partial^4}{\partial x_1^4}-2\,\frac{\partial^4}{c^2\partial x_1^2\partial t_1^2}+\frac{\partial^4}{c^4\partial t_1^4}\right)D^{\mathrm{ret}}(x_2-x_1)\,.$$

By using eq. (2.8) and by the same reasoning as in the foregoing discussion, the terms involving the δ-function on the right-hand side vanish. Therefore we finally get

$$(6.10)\qquad \Delta E_{11}(x_2)=\frac{KM}{8L_1^3}\Delta\varepsilon_x\int_1 \mathrm{d}^4x_1\omega(x_1, x_2)(d_{11}\nabla^2)^2 D^{\mathrm{ret}}(x_2-x_1)\,.$$

(*) See Appendix I.

Taking the weighted average over the space-time region II and signifying by indices (I, II) the disturbance in the region II resulting from the measurement in the region I, we obtain

$$\Delta\bar{E}_{11}^{(\mathrm{I,II})} = \frac{3KM\,\Delta\varepsilon_x}{4L_{\mathrm{I}}^3 L_{\mathrm{II}}^5 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\,\omega(x,x')(d_{11}\nabla^2)^2 D^{\mathrm{ret}}(x'-x)\,. \tag{6.11}$$

By procedures similar to those we have used in deriving $\Delta\bar{E}_{11}^{(\mathrm{I,II})}$, we find for the other uncertainties of electric-type components

$$\left\{\begin{aligned}
\Delta\bar{E}_{22}^{(\mathrm{I,II})} &= \frac{3KM\Delta\varepsilon_x}{4L_{\mathrm{I}}^3 L_{\mathrm{II}}^5 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\,\omega(x,x')(2d_{12}^2 - d_{11}d_{22})\nabla^4 D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\bar{E}_{33}^{(\mathrm{I,II})} &= \frac{3KM\Delta\varepsilon_x}{4L_{\mathrm{I}}^3 L_{\mathrm{II}}^5 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\,\omega(x,x')(2d_{13}^2 - d_{11}d_{33})\nabla^4 D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\bar{E}_{12}^{(\mathrm{I,II})} &= \frac{3KM\Delta\varepsilon_x}{4L_{\mathrm{I}}^3 L_{\mathrm{II}}^5 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\,\omega(x,x')(d_{11}d_{12})\nabla^4 D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\bar{E}_{13}^{(\mathrm{I,II})} &= \frac{3KM\Delta\varepsilon_x}{4L_{\mathrm{I}}^3 L_{\mathrm{II}}^5 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\,\omega(x,x')(d_{11}d_{13})\nabla^4 D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\bar{E}_{23}^{(\mathrm{I,II})} &= \frac{3KM\Delta\varepsilon_x}{4L_{\mathrm{I}}^3 L_{\mathrm{II}}^5 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\,\omega(x,x')(2d_{12}d_{13} - d_{11}d_{23})\nabla^4 D^{\mathrm{ret}}(x'-x)\,.
\end{aligned}\right. \tag{6.12}$$

For the uncertainties of magnetic-type components, we have similarly

$$\left\{\begin{aligned}
\Delta\tilde{H}_{11}^{(\mathrm{I,II})} &= 0\,,\\
\Delta\tilde{H}_{22}^{(\mathrm{I,II})} &= -\frac{KM\Delta\varepsilon_x}{4L_{\mathrm{I}}^3 L_{\mathrm{II}}^3 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\,\omega(x)(d_{12}\nabla^2)\frac{\partial^2}{c\,\partial t\,\partial z} D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\tilde{H}_{33}^{(\mathrm{I,II})} &= \frac{KM\Delta\varepsilon_x}{4L_{\mathrm{I}}^3 L_{\mathrm{II}}^3 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\,\omega(x)(d_{13}\nabla^2)\frac{\partial^2}{c\,\partial t\,\partial y} D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\tilde{H}_{12}^{(\mathrm{I,II})} &= -\frac{KM\Delta\varepsilon_x}{8L_{\mathrm{I}}^3 L_{\mathrm{II}}^3 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\,\omega(x)(d_{11}\nabla^2)\frac{\partial^2}{c\,\partial t\,\partial z} D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\tilde{H}_{13}^{(\mathrm{I,II})} &= \frac{KM\Delta\varepsilon_x}{8L_{\mathrm{I}}^3 L_{\mathrm{I}}^3 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\,\omega(x)(d_{11}\nabla^2)\frac{\partial^2}{c\,\partial t\,\partial y} D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\tilde{H}_{23}^{(\mathrm{I,II})} &= -\frac{KM\Delta\varepsilon_x}{8L_{\mathrm{I}}^3 L_{\mathrm{II}}^3 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\,\omega(x)(2d_{12}\varepsilon_{13l} - d_{11}\varepsilon_{23l})\nabla^2\frac{\partial^2}{c\,\partial t\,\partial x_l} D^{\mathrm{ret}}(x'-x)\,.
\end{aligned}\right. \tag{6.13}$$

On the other hand, from the commutation relations (3.1) we get, for the product of the uncertainties of two weighted averages of electric-type components,

$$
(6.14)\quad \begin{cases}
\Delta\bar{E}_{11}^{(\mathrm{I})}\Delta\bar{E}_{11}^{(\mathrm{II})} \approx \dfrac{9i\hbar K}{L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^5 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x,x')(d_{11}\nabla^2)^2 D^{\mathrm{ret}}(x'-x)\,, \\
\Delta\bar{E}_{11}^{(\mathrm{I})}\Delta\bar{E}_{22}^{(\mathrm{II})} \approx \dfrac{9i\hbar K}{L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^5 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x,x')\,(2d_{12}^2 - d_{11}d_{22})\nabla^4 D^{\mathrm{ret}}(x'-x)\,, \\
\Delta\bar{E}_{11}^{(\mathrm{I})}\Delta\bar{E}_{33}^{(\mathrm{II})} \approx \dfrac{9i\hbar K}{L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^5 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x,x')(2d_{13}^2 - d_{11}d_{33})\nabla^4 D^{\mathrm{ret}}(x'-x)\,, \\
\Delta\bar{E}_{11}^{(\mathrm{I})}\Delta\bar{E}_{12}^{(\mathrm{II})} \approx \dfrac{9i\hbar K}{L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^5 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x,x')(d_{11}d_{12})\nabla^4 D^{\mathrm{ret}}(x'-x)\,, \\
\Delta\bar{E}_{11}^{(\mathrm{I})}\Delta\bar{E}_{13}^{(\mathrm{II})} \approx \dfrac{9i\hbar K}{L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^5 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x,x')(d_{11}d_{13})\nabla^4 D^{\mathrm{ret}}(x'-x)\,, \\
\Delta\bar{E}_{11}^{(\mathrm{I})}\Delta\bar{E}_{23}^{(\mathrm{II})} \approx \dfrac{9i\hbar K}{L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^5 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x,x')(2d_{12}d_{13} - d_{11}d_{23})\nabla^4 D^{\mathrm{ret}}(x'-x)\,.
\end{cases}
$$

Finally from (3.2) we get for the product of uncertainties of the E_{11} average and the averages of magnetic-type components

$$
(6.15)\quad \begin{cases}
\Delta\bar{E}_{11}^{(\mathrm{I})}\Delta\tilde{H}_{11}^{(\mathrm{II})} = 0\,, \\
\Delta\bar{E}_{11}^{(\mathrm{I})}\Delta\tilde{H}_{22}^{(\mathrm{II})} \approx -\dfrac{3i\hbar K}{L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x)(d_{12}\nabla^2)\,\dfrac{\partial^2}{c\,\partial t\,\partial z} D^{\mathrm{ret}}(x'-x)\,, \\
\Delta\bar{E}_{11}^{(\mathrm{I})}\Delta\tilde{H}_{33}^{(\mathrm{II})} \approx \dfrac{3i\hbar K}{L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x)(d_{13}\nabla^2)\,\dfrac{\partial^2}{c\,\partial t\,\partial y} D^{\mathrm{ret}}(x'-x)\,, \\
\Delta\bar{E}_{11}^{(\mathrm{I})}\Delta\tilde{H}_{12}^{(\mathrm{II})} \approx -\dfrac{3i\hbar K}{2L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x)(d_{11}\nabla^2)\,\dfrac{\partial^2}{c\,\partial t\,\partial z} D^{\mathrm{ret}}(x'-x)\,, \\
\Delta\bar{E}_{11}^{(\mathrm{I})}\Delta\tilde{H}_{13}^{(\mathrm{II})} \approx \dfrac{3i\hbar K}{2L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x)(d_{11}\nabla^2)\,\dfrac{\partial^2}{c\,\partial t\,\partial y} D^{\mathrm{ret}}(x'-x)\,, \\
\Delta\bar{E}_{11}^{(\mathrm{I})}\Delta\tilde{H}_{23}^{(\mathrm{II})} \approx -\dfrac{3i\hbar K}{2L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\, \omega(x)(2d_{12}\varepsilon_{13i} - d_{11}\varepsilon_{23i})\nabla^2\,\dfrac{\partial^2}{c\,\partial t\,\partial x_i} D^{\mathrm{ret}}(x'-x).
\end{cases}
$$

Because in this section we have assumed that no light signals from the region II can reach the region I, no disturbances resulting from the measurement in the region II, such as the one we have computed, can influence the measurement in the region I. Hence by comparing eqs. (6.11) and (6.12) with (6.14) and (6.13) with (6.15), we find that $\Delta\bar{E}_{11}^{(\mathrm{I})}$ must take the value $12\hbar/M_{\mathrm{I}}c^2T_{\mathrm{I}}L_{\mathrm{I}}^2\,\Delta\varepsilon_x^{(\mathrm{I})}$ (*). This result does agree completely with the uncertainty of the field strength $\bar{E}_{11}^{(\mathrm{I})}$ given by (4.19), derived from the dynamical equation of motion (4.11) by taking into account the Uncertainty Principle (4.18).

The expressions (6.11), (6.12) and (6.13) relate the disturbances in the region II resulting from the measurement in the region I of the E_{11} weighted average to the uncertainty in the strain $\Delta\varepsilon_x^{(\mathrm{I})}$. Evidently from (6.11), (6.12) and (6.13) it follows that

$$\Delta\bar{E}_{ij}^{(\mathrm{I,II})} = \frac{c^2}{12}\,M_{\mathrm{I}}L_{\mathrm{I}}^2T_{\mathrm{I}}\Delta\varepsilon_x^{(\mathrm{I})}\,\bar{A}_{11ij}^{(\mathrm{I,II})}\,, \tag{6.16}$$

and

$$\begin{cases} \Delta\tilde{H}_{11}^{(\mathrm{I,II})} = 0\,, \\ \Delta\tilde{H}_{ij}^{(\mathrm{I,II})} = \dfrac{c^2}{12}\,M_{\mathrm{I}}L_{\mathrm{I}}^2T_{\mathrm{I}}\Delta\varepsilon_x^{(\mathrm{I})}\,\tilde{B}_{11ij}^{(\mathrm{I,II})} \end{cases} \tag{6.17}$$

where i and j in (6.17) can not take the value 1 at the same time. Recalling the continuous natures of the $\bar{A}$'s and $\bar{B}$'s under the continuous changes of the regions I and II, we see that the uncertainties of field strengths in the region II are simply proportional to the uncertainty of the strain $\Delta\varepsilon_x^{(\mathrm{I})}$ of the first test body. These properties are of decisive importance for the possibility of cancelling the uncontrollable field reactions of the test body by means of suitably chosen force constants for springs joining each test body to its own framework body.

In the foregoing discussion we have considered only the effect of the measurement of the E_{11} weighted field average. In the measurement of $\bar{E}_{22}$ and $\bar{E}_{33}$ we can properly choose a test body which is uniformly stretched not only in the x direction, but also in the y and z directions. On the other hand, the weighted averages of non-diagonal components of the electric type, for example E_{12}, cannot be measured purely by themselves, but are always accompanied by, and mixed with, the measurement of the diagonal components of the electric type, for example E_{11} and E_{22}. This fact can be understood by taking the polarization effect into account, in which E_{12} can be produced by a 45° rotation of either the E_{11} or E_{22} component. Later on we shall return to this

(*) Hereafter we will use the indices I and II to signify explicitly the quantities in the regions I and II.

problem by rotating three principal axes of one test body to arbitrary directions with respect to those of another test body so that their principal axes are no longer parallel to one another.

7. – Magnetic-type field reactions of the test body.

We have obtained the dynamical equation of motion of a spinning test particle in Section 5. We shall consider now a test body of cubical dimensions L which is made up of many uncharged spinning elementary particles and has a uniform spin density $\sigma_x \neq 0$, $\sigma_y = 0$ and $\sigma_z = 0$. Therefore the mass current inside the volume is completely cancelled out, leaving surface currents at $y = \pm L/2$ and $z = \pm L/2$. The components of the energy-momentum tensor of the steady state of the system are thus given by

$$
(7.1)\quad \begin{cases} T_{ij} = 0\,, \\ T_{10} = 0\,, \\ T_{00} = \varrho c^2 \theta\left(x+\frac{L}{2}\right)\theta\left(\frac{L}{2}-x\right)\theta\left(y+\frac{L}{2}\right)\theta\left(\frac{L}{2}-y\right)\theta\left(z+\frac{L}{2}\right)\theta\left(\frac{L}{2}-z\right), \\ T_{20} = \frac{1}{2}\, c\sigma_x \theta\left(x+\frac{L}{2}\right)\theta\left(\frac{L}{2}-x\right)\theta\left(y+\frac{L}{2}\right)\theta\left(\frac{L}{2}-y\right)\left[\delta\left(z+\frac{L}{2}\right)-\delta\left(\frac{L}{2}-z\right)\right], \\ T_{30} = -\frac{1}{2}\, c\sigma_x \theta\left(x+\frac{L}{2}\right)\theta\left(\frac{L}{2}-x\right)\left[\delta\left(y+\frac{L}{2}\right)-\delta\left(\frac{L}{2}-y\right)\right]\theta\left(z+\frac{L}{2}\right)\theta\left(\frac{L}{2}-z\right). \end{cases}
$$

If one shifts the test body in the x direction by an amount Δx at time t_1' and shifts it back at time t_1'', the increases of energy flux ΔT_{20} and ΔT_{30} during the whole time interval are

$$
(7.2)\quad \begin{cases} \Delta T_{20} = \dfrac{c\Delta x}{2}\,\sigma_x \theta(t-t_1')\theta(t_1''-t)\left[\delta\left(x+\frac{L}{2}\right)-\delta\left(\frac{L}{2}-x\right)\right]\cdot \\ \qquad\qquad \cdot\theta\left(y+\frac{L}{2}\right)\theta\left(\frac{L}{2}-y\right)\left[\delta\left(z+\frac{L}{2}\right)-\delta\left(\frac{L}{2}-z\right)\right], \\ \Delta T_{30} = -\dfrac{c\Delta x}{2}\,\sigma_x \theta(t-t_1')\theta(t_1''-t)\left[\delta\left(x+\frac{L}{2}\right)-\delta\left(\frac{L}{2}-x\right)\right]\cdot \\ \qquad\qquad \cdot\left[\delta\left(y+\frac{L}{2}\right)-\delta\left(\frac{L}{2}-y\right)\right]\theta\left(z+\frac{L}{2}\right)\theta\left(\frac{L}{2}-z\right), \end{cases}
$$

and all other components vanish, where $T = t_1'' - t_1'$.

Because during the whole time interval there is an increase of energy flux, there will be a momentum flux at the beginning and end of the time inter-

val T. The components of this flux density are

$$
(7.3)\quad \begin{cases} \Delta T_{21} = \dfrac{c^2\sigma_x}{2}\Delta x\,\theta\left(x+\dfrac{L}{2}\right)\theta\left(\dfrac{L}{2}-x\right)\theta\left(y+\dfrac{L}{2}\right)\theta\left(\dfrac{L}{2}-y\right)\cdot \\ \qquad\qquad \cdot\left[\delta\left(z+\dfrac{L}{2}\right)-\delta\left(\dfrac{L}{2}-z\right)\right]\left[\delta(t-t_{\mathrm{I}}')-\delta(t_{\mathrm{I}}''-t)\right], \\ \Delta T_{31} = -\dfrac{c^2\sigma_x}{2}\Delta x\,\theta\left(x+\dfrac{L}{2}\right)\theta\left(\dfrac{L}{2}-x\right)\cdot \\ \qquad \cdot\left[\delta\left(y+\dfrac{L}{2}\right)-\delta\left(\dfrac{L}{2}-y\right)\right]\theta\left(z+\dfrac{L}{2}\right)\theta\left(\dfrac{L}{2}-z\right)\left[\delta(t-t_{\mathrm{I}}')-\delta(t_{\mathrm{I}}''-t)\right]. \end{cases}
$$

From (7.2) and (7.3) one sees at once that the induced stress components can be written in the form

$$
(7.4)\quad \begin{cases} \Delta T_{20} = \dfrac{\partial^2 P_m}{\partial x\,\partial z}, \\ \Delta T_{30} = -\dfrac{\partial^2 P_m}{\partial x\,\partial y}, \\ \Delta T_{21} = \dfrac{\partial^2 P_m}{\partial z\,\partial x_0}, \\ \Delta T_{31} = -\dfrac{\partial^2 P_m}{\partial y\,\partial x_0}. \end{cases}
$$

with the generalized polarization of the magnetic type

$$
(7.5)\quad P_m = \frac{c\sigma_x}{2}\Delta x\,\theta\left(x+\frac{L}{2}\right)\theta\left(\frac{L}{2}-x\right)\theta\left(y+\frac{L}{2}\right)\theta\left(\frac{L}{2}-y\right)\theta\left(z+\frac{L}{2}\right)\theta\left(\frac{L}{2}-z\right)\cdot \\ \cdot\theta\,(t-t_{\mathrm{I}}')\theta(t_{\mathrm{I}}''-t).
$$

It is easy to verify that the induced stress components as defined above satisfy the conservation law $\partial T_{\mu\nu}/\partial x_\nu = 0$. Hereafter we shall regard $\Delta T_{\mu\nu}$ as the uncertainty of the stress tensor produced by the uncertainty of position Δx.

The uncertainty of the H_{11} components at a space-time point (x_2, y_2, z_2, t_2) due to the uncertainty in position $\Delta x^{(\mathrm{I})}$ of the first test body is, by a calculation similar to the one used in computing the uncertainty of the electric-type component in the previous section,

$$
(7.6)\quad \Delta\tilde{H}_{11}(x_2) = \frac{K\Sigma_x^{(\mathrm{I})}\Delta x^{(\mathrm{I})}e}{4L_{\mathrm{I}}^3}\int_{\mathrm{I}} \mathrm{d}^4x_1 (d_{11}\nabla^2)^2 D^{\mathrm{ret}}(x_2 - x_1),
$$

where $\Sigma_x^{(\mathrm{I})} = \sigma_x L_1^3$ is the total spin angular momentum of the test body covering the spatial region I. By taking the ordinary average over the space-time region II covered by the second test body, we see that

$$\Delta\tilde{H}_{11}^{\mathrm{I,II}} = \frac{K\Sigma_x^{(\mathrm{I})}\Delta x^{(\mathrm{I})}}{4L_{\mathrm{I}}^3 L_{\mathrm{II}}^3 T_{\mathrm{II}}} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x' (d_{11}\nabla^2)^2 D^{\mathrm{ret}}(x'-x)\,. \tag{7.7}$$

Similarly we find for the other uncertainties of magnetic-type components the expressions

$$\left\{\begin{aligned}
\Delta\tilde{H}_{22}^{(\mathrm{I,II})} &= \frac{K\Sigma_x^{(\mathrm{I})}\Delta x^{(\mathrm{I})}}{4L_{\mathrm{I}}^3 L_{\mathrm{II}}^3 T_{\mathrm{II}}} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x' (2d_{12}^2 - d_{11}d_{22})\nabla^4 D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\tilde{H}_{33}^{(\mathrm{I,II})} &= \frac{K\Sigma_x^{(\mathrm{I})}\Delta x^{(\mathrm{I})}}{4L_{\mathrm{I}}^3 L_{\mathrm{II}}^3 T_{\mathrm{II}}} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x' (2d_{13}^2 - d_{11}d_{33})\nabla^4 D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\tilde{H}_{12}^{(\mathrm{I,II})} &= \frac{K\Sigma_x^{(\mathrm{I})}\Delta x^{(\mathrm{I})}}{4L_{\mathrm{I}}^3 L_{\mathrm{II}}^3 T_{\mathrm{II}}} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x' d_{11}d_{12}\nabla^4 D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\tilde{H}_{13}^{(\mathrm{I,II})} &= \frac{K\Sigma_x^{(\mathrm{I})}\Delta x^{(\mathrm{I})}}{4L_{\mathrm{I}}^3 L_{\mathrm{II}}^3 T_{\mathrm{II}}} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x' d_{11}d_{13}\nabla^4 D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\tilde{H}_{23}^{(\mathrm{I,II})} &= \frac{K\Sigma_x^{(\mathrm{I})}\Delta x^{(\mathrm{I})}}{4L_{\mathrm{I}}^3 L_{\mathrm{II}}^3 T_{\mathrm{II}}} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x' (2d_{12}d_{13} - d_{11}d_{23})\nabla^4 D^{\mathrm{ret}}(x'-x)\,.
\end{aligned}\right. \tag{7.8}$$

For the uncertainty in the weighted averages of electric-type components over the space-time region II, we find similarly

$$\left\{\begin{aligned}
\Delta\bar{E}_{11}^{(\mathrm{I,II})} &= 0\,,\\
\Delta\bar{E}_{22}^{(\mathrm{I,II})} &= \frac{3K\Sigma_x^{(\mathrm{I})}\Delta x^{(\mathrm{I})}}{L_{\mathrm{I}}^3 L_{\mathrm{II}}^5 T_{\mathrm{II}}} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x' \omega(x') d_{12}\nabla^2 \frac{\partial^2}{c\,\partial t\,\partial z} D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\bar{E}_{33}^{(\mathrm{I,II})} &= -\frac{3K\Sigma_x^{(\mathrm{I})}\Delta x^{(\mathrm{I})}}{L_{\mathrm{I}}^3 L_{\mathrm{II}}^5 T_{\mathrm{II}}} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x' \omega(x') d_{13}\nabla^2 \frac{\partial^2}{c\,\partial t\,\partial y} D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\bar{E}_{12}^{(\mathrm{I,II})} &= \frac{3K\Sigma_x^{(\mathrm{I})}\Delta x^{(\mathrm{I})}}{2L_{\mathrm{I}}^3 L_{\mathrm{II}}^5 T_{\mathrm{II}}} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x' \omega(x') d_{11}\nabla^2 \frac{\partial^2}{c\,\partial t\,\partial z} D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\bar{E}_{13}^{(\mathrm{I,II})} &= -\frac{3K\Sigma_x^{(\mathrm{I})}\Delta x^{(\mathrm{I})}}{2L_{\mathrm{I}}^3 L_{\mathrm{II}}^5 T_{\mathrm{II}}} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x' \omega(x') d_{11}\nabla^2 \frac{\partial^2}{c\,\partial t\,\partial y} D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\bar{E}_{23}^{(\mathrm{I,II})} &= \frac{3K\Sigma_x^{(\mathrm{I})}\Delta x^{(\mathrm{I})}}{2L_{\mathrm{I}}^3 L_{\mathrm{II}}^5 T_{\mathrm{II}}} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x' \omega(x')(2d_{12}\varepsilon_{13i} - d_{11}\varepsilon_{23i}) \frac{\partial^2}{c\partial t\partial x_i} D^{\mathrm{ret}}(x'-x).
\end{aligned}\right. \tag{7.9}$$

On the other hand, by taking into consideration only the retarded effect from the region I to the region II in (3.16) and (3.17) and by the Uncertainty Principle, we have

$$
(7.10)\quad \left\{
\begin{aligned}
\Delta\tilde{H}^{(\mathrm{I})}_{11}\Delta\tilde{H}^{(\mathrm{II})}_{11} &\approx \frac{i\hbar K}{4L_{\mathrm{I}}^3T_{\mathrm{I}}L_{\mathrm{II}}^3T_{\mathrm{II}}c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'\,(d_{11}\nabla^2)^2D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\tilde{H}^{(\mathrm{I})}_{11}\Delta\tilde{H}^{(\mathrm{II})}_{22} &\approx \frac{i\hbar K}{4L_{\mathrm{I}}^3T_{\mathrm{I}}L_{\mathrm{II}}^3T_{\mathrm{II}}c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'\,(2d_{12}^2-d_{11}d_{22})\nabla^4D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\tilde{H}^{(\mathrm{I})}_{11}\Delta\tilde{H}^{(\mathrm{II})}_{33} &\approx \frac{i\hbar K}{4L_{\mathrm{I}}^3T_{\mathrm{I}}L_{\mathrm{II}}^3T_{\mathrm{II}}c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'\,(2d_{13}^2-d_{11}d_{33})\nabla^4D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\tilde{H}^{(\mathrm{I})}_{11}\Delta\tilde{H}^{(\mathrm{II})}_{12} &\approx \frac{i\hbar K}{4L_{\mathrm{I}}^3T_{\mathrm{I}}L_{\mathrm{II}}^3T_{\mathrm{II}}c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'\,d_{11}d_{12}\nabla^4D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\tilde{H}^{(\mathrm{I})}_{11}\Delta\tilde{H}^{(\mathrm{II})}_{13} &\approx \frac{i\hbar K}{4L_{\mathrm{I}}^3T_{\mathrm{I}}L_{\mathrm{II}}^3T_{\mathrm{II}}c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'\,d_{11}d_{13}\nabla^4D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\tilde{H}^{(\mathrm{I})}_{11}\Delta\tilde{H}^{(\mathrm{II})}_{23} &\approx \frac{i\hbar K}{4L_{\mathrm{I}}^3T_{\mathrm{I}}L_{\mathrm{II}}^3T_{\mathrm{II}}c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'\,(2d_{12}d_{13}-d_{11}d_{23})\nabla^4D^{\mathrm{ret}}(x'-x)\,.
\end{aligned}\right.
$$

and

$$
(7.11)\quad \left\{
\begin{aligned}
\Delta\tilde{H}^{(\mathrm{I})}_{11}\Delta\bar{E}^{(\mathrm{II})}_{11} &\approx 0\,,\\
\Delta\tilde{H}^{(\mathrm{I})}_{11}\Delta\bar{E}^{(\mathrm{II})}_{22} &\approx \frac{i\cdot 3\hbar K}{L_{\mathrm{I}}^3T_{\mathrm{I}}L_{\mathrm{II}}^5T_{\mathrm{II}}c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'\,\omega(x')\,d_{12}\nabla^2\frac{\partial^2}{c\,\partial t\,\partial z}D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\tilde{H}^{(\mathrm{I})}_{11}\Delta\bar{E}^{(\mathrm{II})}_{33} &\approx -\frac{i\cdot 3\hbar K}{L_{\mathrm{I}}^3T_{\mathrm{I}}L_{\mathrm{II}}^5T_{\mathrm{II}}c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'\,\omega(x')\,d_{13}\nabla^2\frac{\partial^2}{c\,\partial t\,\partial y}D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\tilde{H}^{(\mathrm{I})}_{11}\Delta\bar{E}^{(\mathrm{II})}_{12} &\approx \frac{i\cdot 3\hbar K}{2L_{\mathrm{I}}^3T_{\mathrm{I}}L_{\mathrm{II}}^5T_{\mathrm{II}}c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'\,\omega(x')\,d_{11}\nabla^2\frac{\partial^2}{c\,\partial t\,\partial z}D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\tilde{H}^{(\mathrm{I})}_{11}\Delta\bar{E}^{(\mathrm{II})}_{13} &\approx -\frac{i\cdot 3\hbar K}{2L_{\mathrm{I}}^3T_{\mathrm{I}}L_{\mathrm{II}}^5T_{\mathrm{II}}c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'\,\omega(x')\,d_{11}\nabla^2\frac{\partial^2}{c\,\partial t\,\partial y}D^{\mathrm{ret}}(x'-x)\,,\\
\Delta\tilde{H}^{(\mathrm{I})}_{11}\Delta\bar{E}^{(\mathrm{II})}_{23} &\approx \frac{i\cdot 3\hbar K}{2L_{\mathrm{I}}^3T_{\mathrm{I}}L_{\mathrm{II}}^5T_{\mathrm{II}}c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'\,\omega(x')(2d_{12}\varepsilon_{13i}-d_{11}\varepsilon_{23i})\nabla^2\frac{\partial^2}{c\,\partial t\,\partial x_i}\cdot\\
&\qquad\cdot D^{\mathrm{ret}}(x'-x)\,.
\end{aligned}\right.
$$

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Comparing (7.7) and (7.8) with (7.10) and (7.9) with (7.11), we find that $\Delta\tilde{H}_{11}^{(\mathrm{I})}$ must take the value $\hbar/\Sigma_x^{(\mathrm{I})}\Delta x^{(\mathrm{I})}cT_{\mathrm{I}}$. On the other hand, the equation of momentum conservation (5.8) gives the same result upon taking the Uncertainty Principle $\Delta p\,\Delta x\sim\hbar$ into account.

Expressions (7.7), (7.8) and (7.9) can also be written as

$$\Delta\tilde{H}_{ij}^{(\mathrm{I},\mathrm{II})}=c\Sigma_x^{(\mathrm{I})}\Delta x^{(\mathrm{I})}T_{\mathrm{I}}\tilde{O}_{11ij}^{(\mathrm{I},\mathrm{II})} \tag{7.12}$$

and

$$\Delta\bar{E}_{11}^{(\mathrm{I},\mathrm{II})}=0\,,\qquad \Delta\bar{E}_{ij}^{(\mathrm{I},\mathrm{II})}=c\Sigma_x^{(\mathrm{I})}\Delta x^{(\mathrm{I})}T_{\mathrm{I}}\bar{\bar{D}}_{11ij}^{(\mathrm{I},\mathrm{II})}\,, \tag{7.13}$$

where i and j in (7.13) can not take the value 1 at the same time. This shows that the disturbances induced in the region II as a consequence of the measurement of $\tilde{H}_{11}$ in the region I are proportional to the displacement of the position of the first test body. This result plays an important role in the compensation mechanism as was emphasized in reference ([2]).

From the dynamical equation of motion (5.7) we get for the uncertainty of $\tilde{H}_{11}$

$$\Delta\tilde{H}_{11}\sim\frac{\Delta p_x}{c\Sigma_x T}\sim\frac{\hbar}{c\Sigma_x\Delta x T}\,. \tag{7.14}$$

For any Δx, no matter how small, $\Delta\tilde{H}_{11}$ can be measured to an arbitrary accuracy by choosing the total spin Σ_x sufficiently large. We are interested only in the quantum domain, in which

$$\Delta\tilde{H}_{11}\ll u=\frac{L^{\min}}{\sqrt{L^5cT}}\,, \tag{7.15}$$

with the critical field strength u and the minimum length defined by (4.25) and (4.23) respectively. Furthermore, we define a dimensionless factor λ_m by

$$\Delta\tilde{H}_{11}\sim\lambda_m u\,, \tag{7.16}$$

which on being substituted into (7.14) leads to the relation

$$\lambda_m\sim\frac{cL^5}{\Sigma_x\Delta x G}\,u\,. \tag{7.17}$$

The fact that $\lambda_m\ll 1$ and $\Delta x\ll L$ means that the test body must consist of a large number of elementary particles of spin $\hbar/2$. The number of elementary

particles contained in the test body system is given by

$$N \sim \frac{2\Sigma_x}{\hbar} \sim \frac{2}{\lambda_m}\frac{L}{\Delta x}\frac{L}{L_{\min}}\sqrt{\frac{L}{cT}}, \tag{7.18}$$

which is very large for $\lambda_m \ll 1$, $L \gg \Delta x$ and $L \gg L_{\min}$.

8. – Measurement of a single field average.

The measurability of electric-type gravitational field averages has its basis in the fundamental equation of strain-momentum conservation (4.16), which relates the strain momentum of a test body to the field strength. This field strength actually includes all fields in this region: a given gravitational field in the region, a field induced by the self-reaction of the test body covering this region, and induced fields resulting from the measurement of fields in other regions which may include a part of this region.

As a preliminary for a more general discussion of the measurement including two field averages, we shall restrict our discussion simply to the self-reaction of a single field average and then go to that of two field averages. Aside from effects influencing the accurate measurement of the electric-type field strength at the beginning and end of the measuring time interval (as discussed in Section **4**) the strain acceleration of the test body produced by the field itself may cause the test body not to occupy the region where we want to measure the field strength during the measuring time interval. This section affects the accuracy of the measurement. Nevertheless, this effect may be ignored by choosing the mass of the test body large enough.

Referring the eq. (6.16) and making the region I and II coincide, we could separate a given weighted field average $\bar{E}_{11}^{(\mathrm{I})}$, which would be in the region if no field measurement were made, and $\bar{E}_{11}^{(\mathrm{I},\mathrm{I})}$, the self-reaction of the test body resulting from the measurement, in the equation of strain-momentum conservation

$$\pi_x^{(\mathrm{I})''} - \pi_x^{(\mathrm{I})'} = \frac{c^2}{12} M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} (\bar{E}_{11}^{(\mathrm{I})} + \bar{E}_{11}^{(\mathrm{I},\mathrm{I})}) . \tag{8.1}$$

According to the discussion above and that in Section **4**, in order to attain an optimum accuracy of the measurement one has to choose the mass sufficiently large. On the other hand, the heavier the test body is chosen, the larger will be the uncontrollable effect of the self-reaction of the test body, $\Delta\bar{E}_{11}^{(\mathrm{I},\mathrm{I})}$. To reconcile these two opposing requirements, one has to adjust M_{I}, L_{I} and the uncertainty of $\bar{E}_{11}^{(\mathrm{I},\mathrm{I})}$ to give the minimum uncertainty of $\Delta\bar{E}_{11}^{(\mathrm{I})}$.

Since according to (8.1)

$$\Delta\bar{E}_{11}^{(\mathrm{I})} \sim \frac{12\,\Delta\pi_x^{(\mathrm{I})}}{c^2 M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}}} + \Delta\bar{E}_{11}^{(\mathrm{I},\mathrm{I})}\,, \tag{8.2}$$

it follows that, by referring to relation (6.16) and to the uncertainty relation,

$$\Delta\bar{E}_{11}^{(\mathrm{I})} \sim \frac{12\hbar}{c^2 M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} \Delta\varepsilon_x^{(\mathrm{I})}} + \frac{c^2}{12}\, M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} \Delta\varepsilon_x^{(\mathrm{I})} |\bar{A}_{1111}^{(\mathrm{I},\mathrm{I})}|\,. \tag{8.3}$$

From (8.3) the minimum value can be obtained by

$$\Delta_{\min}\bar{E}_{11}^{(\mathrm{I})} \sim \sqrt{\hbar\,|\bar{A}_{1111}^{(\mathrm{I},\mathrm{I})}|}\,, \tag{8.4}$$

which becomes the critical value u in the case $L_{\mathrm{I}} \gg cT_{\mathrm{I}}$ and which may be regarded as an absolute limit on the measurability of gravitational field strengths.

We arrive at the conclusion that the quantum gravitational formalism is applicable only in the classical limiting case, but not in the truly quantum domain. This conclusion can not be justified unless one arranges things so that one can compensate for $\bar{E}_{11}^{(\mathrm{I},\mathrm{I})}$, the uncontrollable self-reaction of the test body, of which the uncertainty is proportional to the unpredictable strain $\Delta\varepsilon_x^{(\mathrm{I})}$; one can retain only the field $\bar{E}_{11}^{(\mathrm{I})}$, which may be compensated for by the measurement except for the inevitable fluctuations. Since we have chosen our test body to be an elastic medium, of which the elastic tension per unit area of the system in the x direction is equal to $-Y\varepsilon_x^{(\mathrm{I})}$, we can attain these conditions by choosing an elastic constant (*) to be

$$Y = \left(\frac{c^2}{12}\, M_{\mathrm{I}} L_{\mathrm{I}}^2\right)^2 T_{\mathrm{I}} \bar{A}_{1111}^{(\mathrm{I},\mathrm{I})}\,, \tag{8.5}$$

as long as the mass density of the test body is large enough and its elastic vibration period is longer than T_{I}. Hence the strain ε_x is much larger than the strain resulting from its oscillation. In the choice of the elastic constant Y above, we note that Y is not only dependent on M_{I} and T_{I}, but also on L_{I}. This is a typical result in the measurement of the electric-type field component and is caused by taking the weighted average of the electric-type field component.

Actually the test body regarded as an elastic medium has to be thought to consist of many component particles which are connected to component particles of a framework body by springs of suitable strength as we have discussed in Section **4**. Therefore we may regard the oscillation period of the framework body to be much longer than the measuring time interval T_{I}. In

(*) The constant Y is actually Young's modulus in the case of the measurement of E_{11}.

order to take retardation effects into account, these springs must be sufficiently short. In the case of a sufficiently heavy test body $\varepsilon_x^{(\mathrm{I})}$ is very small and hence these retardation effects can be neglected.

In the measurement of magnetic-type components, we need also a spring mechanism to compensate its reaction which is proportional to the displacement of the rigid test body as in the measurement of the electromagnetic field strengths.

The spring meachanism for the compensation of self-reactions both in magnetic-type components and electric-type components has no direct relation to quantum concepts but can, in the limit, be only classically describable. We therefore arrive at the conclusion that only the classical description of the field action of the test body gives rise to the interpretability of a single field measurement.

Aside from a single field measurement according to equation (4.16), one may be able to set up a test body which measures only the weighted average of the E_{11} component. It is in principle impossible to measure the ordinary field averages of electric-type components; while the measurement of the weighted averages of magnetic-type components has no physical significance, for only the ordinary averages of magnetic-type components have physical reality according to eq. (5.8).

9. – Measurability of two averages of the same field component.

Following the discussion in the preceding section, we shall study in this section the more general case in which two test bodies covering the regions I and II are so situated that light signals from at least some points of region I can reach region II and vice versa. It is desirable to start from the study of the measurability of two field strengths of the same component, which we shall do in the rest of this section.

By considering the weighted field average $\bar{E}_{11}$ and by neglecting the limitation on the classical describability of the field actions of the test bodies the equations of strain-momentum conservation of two test bodies are as follows:

$$\text{(9.1)} \qquad \begin{cases} \pi_x^{(\mathrm{I})''} - \pi_x^{(\mathrm{I})'} = \dfrac{c^2}{12} M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} (\bar{E}_{11}^{(\mathrm{I})} + \bar{E}_{11}^{(\mathrm{I},\mathrm{I})} + \bar{E}_{11}^{(\mathrm{II},\mathrm{I})}), \\ \pi_x^{(\mathrm{II})''} - \pi_x^{(\mathrm{II})'} = \dfrac{c^2}{12} M_{\mathrm{II}} L_{\mathrm{II}}^2 T_{\mathrm{II}} (\bar{E}_{11}^{(\mathrm{II})} + \bar{E}_{11}^{(\mathrm{II},\mathrm{II})} + \bar{E}_{11}^{(\mathrm{I},\mathrm{II})}), \end{cases}$$

where the uncertainty of $\bar{E}_{11}^{(\mathrm{I},\mathrm{II})}$ is given by (6.16) and $\bar{E}_{11}^{(\mathrm{I},\mathrm{I})}$, $\bar{E}_{11}^{(\mathrm{II},\mathrm{I})}$ by analogous expressions.

According to the discussion in the last section, the self-reaction of each test body can be cancelled by appropriately adjusting spring constants, which are given by expressions analogous to (8.5). By omitting $\bar{E}_{11}^{(\mathrm{I},\mathrm{I})}$ and $\bar{E}_{11}^{(\mathrm{II},\mathrm{II})}$ from (9.1) and referring to (6.16) and (4.18), we have the relations for the uncertainties of the field measurements

$$(9.2)\qquad \begin{cases} \Delta\bar{E}_{11}^{(\mathrm{I})} \sim \dfrac{12\hbar}{c^2 M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} \Delta\varepsilon_x^{(\mathrm{I})}} + \dfrac{c^2}{12} M_{\mathrm{II}} L_{\mathrm{II}}^2 T_{\mathrm{II}} \Delta\varepsilon_x^{(\mathrm{I})} \,|\, \bar{A}_{1111}^{(\mathrm{II},\mathrm{I})} \,|\,, \\ \Delta\bar{E}_{11}^{(\mathrm{II})} \sim \dfrac{12\hbar}{c^2 M_{\mathrm{II}} L_{\mathrm{II}}^2 T_{\mathrm{II}} \Delta\varepsilon_x^{(\mathrm{II})}} + \dfrac{c^2}{12} M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} \Delta\varepsilon_x^{(\mathrm{I})} \,|\, \bar{A}_{1111}^{(\mathrm{I},\mathrm{II})} \,|\,, \end{cases}$$

where we have used the fact that the strains $\varepsilon_x^{(\mathrm{I})}$ and $\varepsilon_x^{(\mathrm{II})}$ are independent of each other and are known with uncertainties $\Delta\varepsilon_x^{(\mathrm{I})}$ and $\Delta\varepsilon_x^{(\mathrm{II})}$. If we simply adjust properly $M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} \Delta\varepsilon_x^{(\mathrm{I})}$ and $M_{\mathrm{II}} L_{\mathrm{II}}^2 T_{\mathrm{II}} \Delta\varepsilon_x^{(\mathrm{II})}$ to fit the minimum value of the product of $\Delta\bar{E}_{11}^{(\mathrm{I})}$ and $\Delta\bar{E}_{11}^{(\mathrm{II})}$ without further detailed consideration of the correlation between subregions of the test bodies, in general, we would arrive at an incorrect result ([13]) to get the sum of the magnitude of $\bar{A}_{1111}^{(\mathrm{I},\mathrm{II})}$ and $\bar{A}_{1111}^{(\mathrm{II},\mathrm{I})}$ in the expression

$$(9.3)\qquad \Delta\bar{E}_{11}^{(\mathrm{I})} \Delta\bar{E}_{11}^{(\mathrm{II})} \sim \hbar[\,|\,\bar{A}_{1111}^{(\mathrm{I},\mathrm{II})}\,| + |\,\bar{A}_{1111}^{(\mathrm{II},\mathrm{I})}\,|\,].$$

This result is in contradiction with expression (3.6). Expression (9.3) agrees with (3.6) only in the case in which one of the quantities $\bar{A}_{1111}^{(\mathrm{I},\mathrm{II})}$ or $\bar{A}_{1111}^{(\mathrm{II},\mathrm{I})}$ vanishes and hence one of the two regions is completely unaffected by the measuring disturbances of the field in the other. Except for this case, they do not agree because (3.6) vanishes even if $\bar{A}_{1111}^{(\mathrm{I},\mathrm{II})}$ and $\bar{A}_{1111}^{(\mathrm{II},\mathrm{I})}$ individually differ from zero in several important cases such as the case that, when T_{I} and T_{II} coincide, $\bar{E}_{11}^{(\mathrm{I})}$ and $\bar{E}_{11}^{(\mathrm{II})}$ can be measured correctly but (9.3) does not vanish.

The agreement between the quantum gravitational formalism and the measurability can be attained by the following more complicated compensation process which is required because of the fact that the strains of both test bodies are not only unknown, but are also entirely independent of each other.

Since the agreement is attained when either $\bar{A}_{1111}^{(\mathrm{I},\mathrm{II})}$ or $\bar{A}_{1111}^{(\mathrm{II},\mathrm{I})}$ vanishes, we need only consider the case where two component particles P_{I} and P_{II} from each of the test bodies I and II influence each other. It is desirable to consider especially the field components over two different space-time regions I and II. Owing to the fact that the retardation of forces admits no direct connection between P_{I} and P_{II} in order to produce a necessary compensation, one can attach to the second test body system another component particle P_{III}

([13]) L. Rosenfeld: *Niels Bohr and Development of Physics* (New York, 1955).

whose distances from the centers of mass of the first and second test body are X_{I} and X_{II} respectively. P_{III} is located in the immediate neighborhood of P_{I} and is joined to P_{I} by a small spring whose spring constant is k_{3a}.

Even though by the exchange of short light signals between P_{II} and P_{III} the total momentum of the second test body system does not change, the total strain momentum of the second test body system does change; consequently we do not know the equation of strain-momentum conservation of the second test body system by the exchange of messages (*) between the subsystems. This fact, however, suggests that a rather complicated process will be necessary. If we denote by $\varepsilon_x^{(\mathrm{III})}$ (**) the strain of P_{III} in the x direction, one can also use the same radiation bundle which measures the strain momentum of the second test body to measure the strain momentum of P_{III} through a more complex apparatus by adjusting the light path appropriately with mirrors as mentioned in Section 4. Indeed, this radiation bundle which measures the strain momentum of P_{II} at the times t'_{II} and t''_{II} has to be arranged to measure the strain momentum of P_{III} at t'_{I} and t''_{I}. By this more complex process one can attain the uncertainty of $\varepsilon_x^{(\mathrm{II})}$ which is exactly the same as that of $\varepsilon_x^{(\mathrm{III})}$.

One may also think that P_{III} is released from the rigid frame at the time t'_{I} and again bound to it at the time t''_{I} so that during T_{I} one can measure the strain momentum transferred to the second test body system. By choosing the force constants of springs joining each test body to its own rigid frame to be k_{1a} and k_{2a}, the strain momentum $-k_{3a}(X_{\mathrm{II}}\,\varepsilon_x^{(\mathrm{III})} - X_{\mathrm{I}}\varepsilon_x^{(\mathrm{II})})T_{\mathrm{I}}X_{\mathrm{I}}$ will be transferred from P_{III} to P_{I}, while $k_{3a}(X_{\mathrm{II}}\,\varepsilon_x^{(\mathrm{III})} - X_{\mathrm{I}}\,\varepsilon_x^{(\mathrm{I})})T_{\mathrm{I}}X_{\mathrm{II}}$ will be transferred from P_{I} to P_{III} during the measuring time interval T_{I}. Then one finds the equation of strain-momentum conservation

$$
(9.4)\quad \begin{cases} \pi_x^{(\mathrm{I})''} - \pi_x^{(\mathrm{I})'} = \dfrac{c^2}{12}\, M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} (\bar{E}_{11}^{(\mathrm{I})} + \bar{E}_{11}^{(\mathrm{I,I})} + \bar{E}_{11}^{(\mathrm{II,I})}) - \\ \qquad\qquad - k_{3a}(X_{\mathrm{II}}\,\varepsilon_x^{(\mathrm{III})} - X_{\mathrm{I}}\varepsilon_x^{(\mathrm{I})})\,T_{\mathrm{I}}X_{\mathrm{I}} - \sum_{\mathrm{I}} k_{1a} x_{\mathrm{I}}^2\, T_{\mathrm{I}} \varepsilon_x^{(\mathrm{I})}, \\ \pi_x^{(\mathrm{II})''} - \pi_x^{(\mathrm{II})'} + \pi_x^{(\mathrm{III})''} - \pi_x^{(\mathrm{III})'} = \dfrac{c^2}{12}\, M_{\mathrm{II}} L_{\mathrm{II}}^2 T_{\mathrm{II}} (\bar{E}_{11}^{(\mathrm{II})} + \bar{E}_{11}^{(\mathrm{II,II})} + \bar{E}_{11}^{(\mathrm{I,II})}) + \\ \qquad\qquad + k_{3a}(X_{\mathrm{II}}\varepsilon_x^{(\mathrm{III})} - X_{\mathrm{I}}\varepsilon_x^{(\mathrm{I})})\,T_{\mathrm{I}}X_{\mathrm{I}} - \sum_{\mathrm{II}} k_{2a} x_{\mathrm{II}}^2\, T_{\mathrm{II}} \varepsilon_x^{(\mathrm{II})}, \end{cases}
$$

where the summations $\sum_{\mathrm{I}}$ and $\sum_{\mathrm{II}}$ are extended over all component particles of each test body. Because one may regard all the springs joining the com-

(*) One may think alternatively that the exchange of messages is allowable only if eqs. (9.5) hold.

(**) This merely means that by receiving a radiation bundle P_{III} suffers the displacement $\varepsilon_x^{(\mathrm{III})} X_{\mathrm{II}}$.

ponent particles of each test body to its own framework body to have the same spring constants, k_{1a} and k_{2a} respectively, we can take k_{1a} and k_{2a} out of the summations. Moreover, replacing $\sum x_1^2$ by the average value $L_1^2/12$, we get

$$(9.5)\quad \begin{cases} \pi_x^{(\mathrm{I})''} - \pi_x^{(\mathrm{I})'} = \dfrac{c^2}{12} M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} (\bar{E}_{11}^{(\mathrm{I})} + \bar{E}_{11}^{(\mathrm{I,I})} + \bar{E}_{11}^{(\mathrm{II,I})}) - \\ \qquad - \left(\dfrac{L_{\mathrm{I}}^2}{12} k_{1a} - k_{3a} X_{\mathrm{I}}^2 \right) T_{\mathrm{I}}\, \varepsilon_x^{(\mathrm{I})} - k_{3a} X_{\mathrm{I}} X_{\mathrm{II}} T_{\mathrm{I}} \varepsilon_x^{(\mathrm{III})}, \\ \pi_x^{(\mathrm{II})''} - \pi_x^{(\mathrm{II})'} + \pi_x^{(\mathrm{III})''} - \pi_x^{(\mathrm{III})'} = \dfrac{c^2}{12} M_{\mathrm{II}} L_{\mathrm{II}}^2 T_{\mathrm{II}} (\bar{E}_{11}^{(\mathrm{II})} + \bar{E}_{11}^{(\mathrm{II,II})} + \bar{E}_{11}^{(\mathrm{I,II})}) - \\ \qquad - \left(\dfrac{L_{\mathrm{II}}^2}{12} k_{2a} T_{\mathrm{II}} \varepsilon_x^{(\mathrm{II})} - k_{3a} X_{\mathrm{II}}^2 T_{\mathrm{I}} \varepsilon_x^{(\mathrm{III})} \right) - k_{3a} X_{\mathrm{I}} X_{\mathrm{II}} T_{\mathrm{I}} \varepsilon_x^{(\mathrm{I})}. \end{cases}$$

We shall now consider the case of partially overlapping regions. There are three kinds of adjustable springs; namely, one of them connecting the component particles of the first test body with those of the second test body, and the other two joining the component particles of each test body to its own framework body. The former kind is confined only to the overlapping spatial region and is attached only during the overlapping time interval. Hence in this case we obtain

$$(9.6)\quad \begin{cases} \pi_x^{(\mathrm{I})''} - \pi_x^{(\mathrm{I})'} = \dfrac{c^2}{12} M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} (\bar{E}_{11}^{(\mathrm{I})} + \bar{E}_{11}^{(\mathrm{I,I})} + \bar{E}_{11}^{(\mathrm{II,I})}) - \varkappa_{1a} \varepsilon_x^{(\mathrm{I})} - \varkappa_{3a} \varepsilon_x^{(\mathrm{III})}, \\ \pi_x^{(\mathrm{II})''} - \pi_x^{(\mathrm{II})'} = \dfrac{c^2}{12} M_{\mathrm{II}} L_{\mathrm{II}}^2 T_{\mathrm{II}} (\bar{E}_{11}^{(\mathrm{II})} + \bar{E}_{11}^{(\mathrm{II,II})} + \bar{E}_{11}^{(\mathrm{I,II})}) - \varkappa_{2a} \varepsilon_x^{(\mathrm{II})} - \varkappa_{3a} \varepsilon_x^{(\mathrm{I})}, \end{cases}$$

where $\varkappa_{1a}$, $\varkappa_{2a}$ and $\varkappa_{3a}$ are dependent on the geometry of each test body and in addition are functions of three independent adjustable spring constants k_{1a}, k_{2a} and k_{3a}.

We now turn to eq. (9.5). We choose the spring constants

$$(9.7)\quad \begin{cases} k_{3a} = \dfrac{c^4}{288 X_{\mathrm{I}} X_{\mathrm{II}}} M_{\mathrm{I}} M_{\mathrm{II}} L_{\mathrm{I}}^2 L_{\mathrm{II}}^2 T_{\mathrm{II}} (\bar{A}_{1111}^{(\mathrm{I,II})} + \bar{A}_{1111}^{(\mathrm{II,I})}), \\ k_{1a} = \dfrac{c^2}{12} M_{\mathrm{I}}^2 L_{\mathrm{I}}^2 T_{\mathrm{I}} \bar{A}_{1111}^{(\mathrm{I,I})} + \dfrac{c^4}{24} M_{\mathrm{I}} M_{\mathrm{II}} T_{\mathrm{II}} L_{\mathrm{II}}^2 \left\{ \left(1 + \dfrac{X_{\mathrm{I}}}{X_{\mathrm{II}}} \right) (\bar{A}_{1111}^{(\mathrm{I,II})} + \bar{A}_{1111}^{(\mathrm{II,I})}) \right\}, \end{cases}$$

and an analogous expression for k_{2a} by interchanging the indices I and II in

the expression of k_{1a}. By virtue of (6.16) we get

$$(9.8)\quad \begin{cases} \pi_x^{(\mathrm{I})''} - \pi_x^{(\mathrm{I})'} = \dfrac{c^2}{12} M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} \bar{E}_{11}^{(\mathrm{I})} + \dfrac{c^4}{288} M_{\mathrm{I}} M_{\mathrm{II}} L_{\mathrm{I}}^2 L_{\mathrm{II}}^2 T_{\mathrm{I}} T_{\mathrm{II}} \{ -\varepsilon_x^{(\mathrm{II})} (\bar{A}_{1111}^{(\mathrm{I,II})} - \bar{A}_{1111}^{(\mathrm{II,I})}) \}, \\ \pi_x^{(\mathrm{II})''} - \pi_x^{(\mathrm{II})'} + \pi_x^{(\mathrm{III})''} - \pi_x^{(\mathrm{III})'} = \dfrac{c^2}{12} M_{\mathrm{II}} L_{\mathrm{II}}^2 T_{\mathrm{II}} \bar{E}_{11}^{(\mathrm{II})} + \\ \qquad + \dfrac{c^4}{288} M_{\mathrm{I}} M_{\mathrm{II}} L_{\mathrm{I}}^2 L_{\mathrm{II}}^2 T_{\mathrm{I}} T_{\mathrm{II}} \{ \varepsilon_x^{(\mathrm{I})} (\bar{A}_{1111}^{(\mathrm{I,II})} - \bar{A}_{1111}^{(\mathrm{II,I})}) \} \end{cases}$$

We therefore obtain two uncertainties

$$(9.9)\quad \begin{cases} \Delta \bar{E}_{11}^{(\mathrm{I})} \sim \dfrac{12\hbar}{c^2 M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} \Delta \varepsilon_x^{(\mathrm{I})}} + \dfrac{c^2}{288} M_{\mathrm{II}} L_{\mathrm{II}} T_{\mathrm{II}} \Delta \varepsilon_x^{(\mathrm{II})} | \bar{A}_{1111}^{(\mathrm{I,II})} - \bar{A}_{1111}^{(\mathrm{II,I})} |, \\ \Delta \bar{E}_{11}^{(\mathrm{II})} \sim \dfrac{12\hbar}{c^2 M_{\mathrm{II}} L_{\mathrm{II}}^2 T_{\mathrm{II}} \Delta \varepsilon_x^{(\mathrm{II})}} + \dfrac{c^2}{288} M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} \Delta \varepsilon_x^{(\mathrm{I})} | \bar{A}_{1111}^{(\mathrm{I,II})} - \bar{A}_{1111}^{(\mathrm{II,I})} |, \end{cases}$$

from which, according to a process similar to that discussed in the previous section, we find the minimum value of the product of two uncertainties

$$(9.10)\qquad \Delta \bar{E}_{11}^{(\mathrm{I})} \, \Delta \bar{E}_{11}^{(\mathrm{II})} \sim \hbar | \bar{A}_{1111}^{(\mathrm{I,II})} - \bar{A}_{1111}^{(\mathrm{II,I})} |.$$

(9.10) is in complete agreement with the first expression of (3.6) which has resulted from the quantum gravitational formalism.

In the case of two test bodies having a partially overlapping region, we can as well attain the result (9.10). However, in this case we have to connect springs joining the component particles of different test bodies to each other only during the overlapping time interval, and consequently $\varkappa_{1a}$, $\varkappa_{2a}$ and $\varkappa_{3a}$ in eqs. (9.6) are also functions of this overlapping time interval.

10. – Measurability of two averages of different field components.

When either of the field components is of the magnetic type, the measurability is completely the same as that described in reference [2]. Hence we shall discuss merely three cases: when either of the field components is of the electric type, when one component is of the electric type and the rest are of the magnetic type. To elucidate this statement we shall study the measurement of the weighted average of E_{11} over the region I and the weighted average of E_{22} or the ordinary average of H_{11} (or H_{22}) over the region II.

Let us start with the first case, namely, when either of the field components is of the electric type. For either of the test bodies covering the space-time regions I and II respectively, the equations of strain-momentum conservation are given by

$$\begin{cases} \pi_x^{(\mathrm{I})''} - \pi_x^{(\mathrm{I})'} = \dfrac{c^2}{12} M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} (\bar{E}_{11}^{(\mathrm{I})} + \bar{E}_{11}^{(\mathrm{I,I})} + \bar{E}_{11}^{(\mathrm{II,I})}), \\ \pi_y^{(\mathrm{II})''} - \pi_y^{(\mathrm{II})'} = \dfrac{c^2}{12} M_{\mathrm{II}} L_{\mathrm{II}}^2 T_{\mathrm{II}} (\bar{E}_{22}^{(\mathrm{II})} + \bar{E}_{22}^{(\mathrm{II,II})} + \bar{E}_{22}^{(\mathrm{I,II})}) . \end{cases} \tag{10.1}$$

If we do not take into account the correlation between a component particle of the first test body and one in the second test body, by suitably adjusting the spring constants for springs joining each test body to its own rigid frame we have, by a calculation similar to that used in deriving (9.3) from (9.1),

$$\Delta \bar{E}_{11}^{(\mathrm{I})} \Delta \bar{E}_{22}^{(\mathrm{II})} \sim \hbar [| \bar{A}_{1122}^{(\mathrm{I,II})} | + | \bar{A}_{1122}^{(\mathrm{II,I})} |] \tag{10.2}$$

which is, in general, in contradiction with the results from the quantum gravitational formalism and agrees with it only in special cases in which either $\bar{A}_{1122}^{(\mathrm{I,II})}$ or $\bar{A}_{1122}^{(\mathrm{II,I})}$ vanishes. The result of the formalism requires that when two time intervals T_{I} and T_{II} coincide the different components of the same type of field (*), say $\bar{E}_{11}^{(\mathrm{I})}$ and $\bar{E}_{22}^{(\mathrm{II})}$, can be measured correctly, but (10.2) does not vanish and hence disagrees with the formalism.

Now we choose from each of the test bodies I and II two component particles, P_{I} and P_{II}. Two radiation bundles of photons are arranged to measure the x component of the strain momentum of the first test body and the y component of the strain momentum of the second test body. We can, as before, choose another component particle P_{III}, attaching it to the second test body system, to be located in the immediate neighborhood of P_{I}. P_{III} is joined to P_{I} through a lever system which consists of two equal and perpendicular arms, one each in the x and y directions, and is rotatable about its joint. The joint of the lever is bound to the rigid frame. P_{I} is connected to the end of the x arm of the lever with the help of a spring which is itself in the y direction, while P_{III} is connected to the end of the y arm of the lever through another spring, acting in the x direction and having the same force constant k_{3b} and length as that of the first spring employed in connecting P_{I}.

We use the same radiation bundle which measures the strain momentum of the second test body at times t_{II}' and t_{II}'' and also P_{III} at times t_{I}' and t_{I}'' so that the uncertainty of $\varepsilon_y^{(\mathrm{III})}$ is exactly the same as that of $\varepsilon_y^{(\mathrm{II})}$. If we denote by X_{I} and Y_{II} the distances of P_{I} and P_{III} from the center of mass of each test body,

(*) The case discussed in (3.18).

then during the time interval T_{I} an amount of strain momentum $k_{3b}(X_{\mathrm{II}}\varepsilon_y^{(\mathrm{III})} - -X_{\mathrm{I}}\varepsilon_x^{(\mathrm{I})})T_{\mathrm{I}}X_{\mathrm{I}}$ will be transferred from P_{III} to P_{I} and an amount $-k_{3b}(Y_{\mathrm{II}}\varepsilon_y^{(\mathrm{III})} - -X_{\mathrm{I}}\varepsilon_x^{(\mathrm{I})})T_{\mathrm{I}}Y_{\mathrm{II}}$ from P_{I} to P_{III}. The equations of strain-momentum conservation for each test body system are then

$$
(10.3)\quad \begin{cases} \pi_x^{(\mathrm{I})''} - \pi_x^{(\mathrm{I})'} = \dfrac{c^2}{12} M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} (\bar{E}_{11}^{(\mathrm{I})} + \bar{E}_{11}^{(\mathrm{I,I})} + \bar{E}_{11}^{(\mathrm{II,I})}) + \\ \qquad + k_{3b}(Y_{\mathrm{II}}\varepsilon_y^{(\mathrm{III})} - X_{\mathrm{I}}\varepsilon_y^{(\mathrm{I})})T_{\mathrm{I}}X_{\mathrm{I}} - \sum_{\mathrm{I}} k_{1b} x_1^2 T_{\mathrm{I}} \varepsilon_x^{(\mathrm{I})}, \\ \pi_y^{(\mathrm{II})''} - \pi_y^{(\mathrm{II})'} + \pi_y^{(\mathrm{III})''} - \pi_y^{(\mathrm{III})'} = \dfrac{c^2}{12} M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} (\bar{E}_{22}^{(\mathrm{II})} + \bar{E}_{22}^{(\mathrm{II,II})} + \bar{E}_{22}^{(\mathrm{I,II})}) - \\ \qquad - k_{3b}(Y_{\mathrm{II}}\varepsilon_y^{(\mathrm{III})} - X_{\mathrm{I}}\varepsilon_y^{(\mathrm{I})})T_{\mathrm{I}}Y_{\mathrm{II}} - \sum_{\mathrm{II}} k_{2b} y_2^2 T_{\mathrm{II}} \varepsilon_y^{(\mathrm{II})}. \end{cases}
$$

By referring to expression (6.16) and using suitably chosen spring constants k_{1b}, k_{2b} and k_{3b}, we have

$$
(10.4)\quad \begin{cases} \pi_x^{(\mathrm{I})''} - \pi_x^{(\mathrm{I})'} = \dfrac{c^2}{12} M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} \bar{E}_{11}^{(\mathrm{I})} + \dfrac{c^4}{288} M_{\mathrm{I}} M_{\mathrm{II}} L_{\mathrm{I}}^2 L_{\mathrm{II}}^2 T_{\mathrm{I}} T_{\mathrm{II}}{}^{r} - \varepsilon_y^{(\mathrm{I})}(\bar{A}_{1122}^{(\mathrm{I,II})} - \bar{A}_{1122}^{(\mathrm{II,I})})], \\ \pi_y^{(\mathrm{II})''} - \pi_y^{(\mathrm{II})'} + \pi_y^{(\mathrm{III})''} - \pi_y^{(\mathrm{III})'} = \dfrac{c^2}{12} M_{\mathrm{II}} L_{\mathrm{II}}^2 T_{\mathrm{II}} \bar{E}_{22}^{(\mathrm{II})} + \\ \qquad + \dfrac{c^4}{288} M_{\mathrm{I}} M_{\mathrm{II}} L_{\mathrm{I}}^2 L_{\mathrm{II}}^2 T_{\mathrm{I}} T_{\mathrm{II}} [\varepsilon_x^{(\mathrm{I})}(\bar{A}_{1122}^{(\mathrm{I,II})} - \bar{A}_{1122}^{(\mathrm{II,I})})], \end{cases}
$$

from which follow the uncertainties of field averages in each region

$$
(10.5)\quad \begin{cases} \Delta\bar{E}_{11}^{(\mathrm{I})} \sim \dfrac{12\hbar}{c^2 M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} \Delta\varepsilon_x^{(\mathrm{I})}} + \dfrac{c^2}{24} M_{\mathrm{II}} L_{\mathrm{II}}^2 T_{\mathrm{II}} \Delta\varepsilon_y^{(\mathrm{II})} |\bar{A}_{1122}^{(\mathrm{I,II})} - \bar{A}_{1122}^{(\mathrm{II,I})}|, \\ \Delta\bar{E}_{22}^{(\mathrm{II})} \sim \dfrac{12\hbar}{c^2 M_{\mathrm{II}} L_{\mathrm{II}}^2 T_{\mathrm{II}} \Delta\varepsilon_y^{(\mathrm{II})}} + \dfrac{c^2}{24} M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} \Delta\varepsilon_x^{(\mathrm{I})} |\bar{A}_{1122}^{(\mathrm{I,II})} - \bar{A}_{1122}^{(\mathrm{II,I})}|, \end{cases}
$$

and furthermore the minimum value of the product of uncertainties is

$$
(10.6)\qquad \Delta\bar{E}_{11}^{(\mathrm{I})} \Delta\bar{E}_{22}^{(\mathrm{II})} \sim \hbar |\bar{A}_{1122}^{(\mathrm{I,II})} - \bar{A}_{1122}^{(\mathrm{II,I})}|,
$$

which is in harmony with the result of quantum gravitodynamics.

For the partially overlapping spatial region a discussion can be carried out as has been done after expression (9.6) and an agreement can be similarly attained.

Now let us consider a simpler case: the measurability of E_{11} and H_{11}. The equations of strain-momentum and momentum conservation are given by

$$\text{(10.7)} \qquad \begin{cases} \pi_x^{(\mathrm{I})''} - \pi_x^{(\mathrm{I})'} = \dfrac{c^2}{12} M_{\mathrm{I}} L_1^2 T_{\mathrm{I}} (\bar{E}_{11}^{(\mathrm{I})} + \bar{E}_{11}^{(\mathrm{I,II})} + \bar{E}_{11}^{(\mathrm{II,I})}) , \\ p_x^{(\mathrm{II})''} - p_x^{(\mathrm{II})'} = c \Sigma_x^{(\mathrm{II})} T_{\mathrm{II}} (\tilde{H}_{11}^{(\mathrm{II})} + \tilde{H}_{11}^{(\mathrm{II,II})} + \tilde{H}_{11}^{(\mathrm{I,II})}) , \end{cases}$$

where $\Delta\bar{E}_{11}^{(\mathrm{I,II})}$, etc., are weighted averages while $\Delta\tilde{H}_{11}^{(\mathrm{I,II})}$, etc., are ordinary averages as defined by (7.12). We get $\Delta\tilde{H}_{11}^{(\mathrm{II,II})}$ from the corresponding equation by making I and II coincide. The self-reactions of $\bar{E}_{11}^{(\mathrm{I,I})}$ can be cancelled as before by choosing $k_{1b} = c^2 M_{\mathrm{I}} \bar{A}_{1111}^{(\mathrm{I,I})} / L_{\mathrm{I}}$ for the force constant of springs joining the component particles of the first test body to its own framework body. The self-reaction $\tilde{H}_{11}^{(\mathrm{II,II})}$ resulting from $x^{(\mathrm{II})}$, the displacement in the x direction of the second test body system (which is considered to be a rigid body), can also be compensated for during the interval of measurement T_{II} by binding the component particles of the second test body to the fixed frame through springs having the same force constant k_{2b}, which is chosen to be $c^2 \Sigma_x^{(\mathrm{II})^2} T_{\mathrm{II}} \tilde{C}_{1111}^{(\mathrm{II,II})} / V_{\mathrm{II}}$ with $\tilde{C}_{1111}^{(\mathrm{II,II})}$ given by an expression analogous to (3.14), so that the force exerted on the second test body is equal to $-c^2 \Sigma_x^{(\mathrm{II})^2} T_{\mathrm{II}} \tilde{C}_{1111}^{(\mathrm{II,II})} x^{(\mathrm{II})}$. Omitting each self-reaction $\bar{E}_{11}^{(\mathrm{I,I})}$ and $\tilde{H}_{11}^{(\mathrm{II,II})}$ from each equation of (10.7) and referring to (6.17) and (7.13), according to which $\bar{E}_{11}^{(\mathrm{II,I})}$ and $\tilde{H}_{11}^{(\mathrm{I,II})}$ vanish, we find

$$\text{(10.8)} \qquad \begin{cases} \pi_x^{(\mathrm{I})''} - \pi_x^{(\mathrm{I})'} = \dfrac{c^2}{12} M_{\mathrm{I}} L_1^2 T_{\mathrm{I}} \bar{E}_{11}^{(\mathrm{I})} , \\ p_x^{(\mathrm{II})''} - p_x^{(\mathrm{II})'} = c \Sigma_x^{(\mathrm{II})} T_{\mathrm{II}} \tilde{H}_{11}^{(\mathrm{II})} , \end{cases}$$

from which the minimum value of the product of uncertainties is immediately given by

$$\text{(10.9)} \qquad \Delta\bar{E}_{11}^{(\mathrm{I})} \, \Delta\tilde{H}_{11}^{(\mathrm{II})} \sim 0 ,$$

in agreement with expression (3.11) derived from the formalism.

Let us finally consider the measurability of E_{11} and H_{22}: The equations of strain-momentum and momentum conservation are

$$\text{(10.11)} \qquad \begin{cases} \pi_x^{(\mathrm{I})''} - \pi_x^{(\mathrm{I})'} = \dfrac{c^2}{12} M_{\mathrm{I}} L_1^2 T_{\mathrm{I}} (\bar{E}_{11}^{(\mathrm{I})} + \bar{E}_{11}^{(\mathrm{I,I})} + \bar{E}_{11}^{(\mathrm{II,I})}) , \\ p_y^{(\mathrm{II})''} - p_y^{(\mathrm{II})'} = c \Sigma_y^{(\mathrm{II})} T_{\mathrm{II}} (\tilde{H}_{22}^{(\mathrm{II})} + \tilde{H}_{22}^{(\mathrm{II,II})} + \tilde{H}_{22}^{(\mathrm{I,II})}) . \end{cases}$$

As previously, if we join only the component particles of each test body to its own framework body by springs of suitably chosen force constants k_{1c} and k_{2c}, we get

$$(10.11)\qquad \begin{cases} \pi_x^{(\mathrm{I})''} - \pi_x^{(\mathrm{I})'} = \dfrac{c^2}{12} M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} (\bar{\bar{E}}_{11}^{(\mathrm{I})} + c\Sigma_y^{(\mathrm{II})} T_{\mathrm{II}} y^{(\mathrm{II})} \bar{\bar{B}}_{1122}^{(\mathrm{II},\mathrm{I})}) , \\ p_y^{(\mathrm{II})''} - p_y^{(\mathrm{II})'} = c\Sigma_y^{(\mathrm{II})} T_{\mathrm{II}} \left(\tilde{\tilde{H}}_{22}^{(\mathrm{II})} + \dfrac{c^2}{12} M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} \varepsilon_x^{(\mathrm{I})} \tilde{\tilde{B}}_{1122}^{(\mathrm{I},\mathrm{II})} \right) , \end{cases}$$

where $\tilde{\tilde{B}}_{1122}^{(\mathrm{I},\mathrm{II})}$ is defined by (3.9) and $\bar{\bar{B}}_{1122}^{(\mathrm{II},\mathrm{I})}$ is defined by taking the ordinary average value over the space-time region II and the weighted average value over the space-time region I as

$$(10.12)\qquad \bar{\bar{B}}_{1122}^{(\mathrm{II},\mathrm{I})} = \frac{-3K}{L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}}} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x' \, \omega(x)(d_{12}\nabla^2) \frac{\partial^2}{c\,\partial t\,\partial z} D^{\mathrm{ret}}(x-x').$$

From (10.11) it immediately follows that

$$(10.13)\qquad \Delta\bar{\bar{E}}_{11}^{(\mathrm{I})} \Delta\tilde{\tilde{H}}_{22}^{(\mathrm{II})} \sim \hbar \left[|\bar{\bar{B}}_{1122}^{(\mathrm{II},\mathrm{I})}| + |\tilde{\tilde{B}}_{1122}^{(\mathrm{I},\mathrm{II})}| \right],$$

which again, in general, disagrees with the result from the formalism. From expression (3.7) of the formalism $\Delta\bar{E}_{11}\Delta\bar{H}_{22}$ vanishes when two volumes V_{I} and V_{II} coincide even if $\bar{\bar{B}}_{1122}^{(\mathrm{II},\mathrm{I})}$ and $\tilde{\tilde{B}}_{1122}^{(\mathrm{I},\mathrm{II})}$ individually do not vanish but (10.13) differs from zero.

The agreement can be attained as follows: as in the foregoing discussion we choose from each of the test bodies I and II component particles P_{I} and P_{II} respectively. To the second test body is attached another component particle P_{III} whose momentum is measured at times t_{I}' and t_{I}'' by means of the same radiation bundle which measures the momentum of the second test body at times t_{II}' and t_{II}'', so that the uncertainty of the position of P_{III} is exactly the same as that of P_{II}. We join P_{III} through a spring acting in the y direction and having the force constant k_{3c} with the end of the x arm of the lever as used in a previous discussion of this section. P_{I} is connected to the end of the y arm of the lever with the help of a spring which acts itself in the x direction and which has the same length and force constant as that of the first spring. During the time interval T_{I} the strain momentum $-k_{3c}(y^{(\mathrm{III})} - \varepsilon_x^{(\mathrm{I})} X_{\mathrm{I}}) X_{\mathrm{I}} T_{\mathrm{I}}$ will be transferred from P_{III} to P_{I}, while the momentum $k_{3c}(y^{(\mathrm{III})} - \varepsilon_x^{(\mathrm{I})} X_{\mathrm{I}}) T_{\mathrm{I}}$ will be transferred from P_{I} to P_{III}, where X_{I} is the distance of P_{I} from the center of mass of the first test body system. Regarding P_{III} as part of the second test body

system, we have

$$
(10.14)\quad \begin{cases} \pi_x^{(\mathrm{I})''} - \pi_x^{(\mathrm{I})'} = \dfrac{c^2}{12} M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} (\bar{E}_{11}^{(\mathrm{I})} + \bar{E}_{11}^{(\mathrm{I},\mathrm{I})} + \bar{E}_{11}^{(\mathrm{II},\mathrm{I})}) - k_{3c} y^{(\mathrm{III})} X_{\mathrm{I}} T_{\mathrm{I}} - \\ \qquad\qquad - \left(- k_{3c} X_{\mathrm{I}}^2 + k_{1c} \dfrac{L_{\mathrm{I}}^2}{12} \right) \varepsilon_x^{(\mathrm{I})} T_x , \\ p_y^{(\mathrm{II})''} - p_y^{(\mathrm{II})'} + p_y^{(\mathrm{III})''} - p_y^{(\mathrm{III})'} = \\ = c \Sigma_y^{(\mathrm{II})} T_{\mathrm{II}} (\tilde{H}_{22}^{(\mathrm{II})} + \tilde{H}_{22}^{(\mathrm{II},\mathrm{II})} + \tilde{H}_{22}^{(\mathrm{I},\mathrm{II})}) - k_{3c} \varepsilon_x^{(\mathrm{I})} X_{\mathrm{I}} T_{\mathrm{I}} - (- k_{3c} y^{(\mathrm{III})} T_{\mathrm{I}} + k_{2c} y^{(\mathrm{II})} T_{\mathrm{II}}) . \end{cases}
$$

We now choose the spring constants

$$
(10.15)\quad \begin{cases} k_{1c} = c^3 \Sigma_y^{(\mathrm{II})} M_{\mathrm{I}} X_{\mathrm{I}} \bar{\tilde{D}}_{2211}^{(\mathrm{II},\mathrm{I})} + \dfrac{c^4}{24} M_{\mathrm{I}} M_{\mathrm{II}} T_{\mathrm{II}} L_{\mathrm{II}}^2 (\bar{A}_{1111}^{(\mathrm{I},\mathrm{II})} + \bar{A}_{1111}^{(\mathrm{II},\mathrm{I})}) , \\ k_{2c} = \dfrac{c^3}{12 X_{\mathrm{I}}} \Sigma_y^{(\mathrm{II})} M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} \bar{\tilde{D}}_{2211}^{(\mathrm{II},\mathrm{I})} + \dfrac{c}{2} \Sigma_y^{(\mathrm{II})} T_{\mathrm{II}} (\tilde{C}_{2222}^{(\mathrm{I},\mathrm{II})} + \tilde{C}_{2222}^{(\mathrm{II},\mathrm{I})}) , \\ k_{3c} = \dfrac{c^3}{12 X_{\mathrm{I}}} \Sigma_y^{(\mathrm{II})} M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{II}} \bar{\tilde{D}}_{2211}^{(\mathrm{II},\mathrm{I})} , \end{cases}
$$

where

$$
(10.16)\quad \begin{cases} \bar{\tilde{D}}_{2211}^{(\mathrm{II},\mathrm{I})} = \dfrac{3K}{L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c} \int_{\mathrm{I}} \mathrm{d}^4 x \int_{\mathrm{II}} \mathrm{d}^4 x' \, \omega(x) \, d_{12} \nabla^2 \dfrac{\partial^2}{c \, \partial t \, \partial z} D^{\mathrm{ret}}(x - x') , \\ \tilde{C}_{2222}^{(\mathrm{I},\mathrm{II})} = \dfrac{K}{4 L_{\mathrm{I}}^3 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c} \int_{\mathrm{I}} \mathrm{d}^4 x \int_{\mathrm{II}} \mathrm{d}^4 x' (d_{22} \nabla^2)^2 D^{\mathrm{ret}}(x' - x) . \end{cases}
$$

Equations (10.14) now take the form

$$
(10.17)\quad \begin{cases} \pi_x^{(\mathrm{I})''} - \pi_x^{(\mathrm{I})'} = \dfrac{c^2}{12} M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} \bar{E}_{11}^{(\mathrm{I})} + \dfrac{c^3}{24} M_{\mathrm{I}} L_{\mathrm{I}}^2 \Sigma_y^{(\mathrm{II})} T_{\mathrm{I}} T_{\mathrm{II}} [- y^{(\mathrm{II})} (\bar{\tilde{B}}_{1122}^{(\mathrm{I},\mathrm{II})} - \bar{\tilde{B}}_{1122}^{(\mathrm{II},\mathrm{I})})] , \\ p_y^{(\mathrm{II})''} - p_y^{(\mathrm{II})'} + p_y^{(\mathrm{III})''} - p_y^{(\mathrm{III})'} = c \Sigma_y^{(\mathrm{II})} T_{\mathrm{II}} \tilde{H}_{22}^{(\mathrm{II})} + \dfrac{c^3}{24} M_{\mathrm{I}} L_{\mathrm{I}}^2 \Sigma_y^{(\mathrm{II})} T_{\mathrm{I}} T_{\mathrm{II}} [\varepsilon_x^{(\mathrm{I})} (\bar{\tilde{B}}_{1122}^{(\mathrm{I},\mathrm{II})} - \bar{\tilde{B}}_{1122}^{(\mathrm{II},\mathrm{I})})] . \end{cases}
$$

Hence the unknown strain in the x direction in the first equation of (10.14) can be completely compensated for by these suitably chosen force constants of the spring mechanism. The unknown displacement in the second equation of (10.14) is similarly compensated for. The uncertainties of the field averages are evidently

$$
(10.18)\quad \begin{cases} \Delta \bar{E}_{11}^{(\mathrm{I})} \sim \dfrac{12 \hbar}{c^2 M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} \Delta \varepsilon_x^{(\mathrm{I})}} + \dfrac{c}{2} \Sigma_y^{(\mathrm{II})} T_{\mathrm{II}} \Delta y^{(\mathrm{II})} | \bar{\tilde{B}}_{1122}^{(\mathrm{I},\mathrm{II})} - \bar{\tilde{B}}_{1122}^{(\mathrm{II},\mathrm{I})} | , \\ \Delta \tilde{H}_{22}^{(\mathrm{II})} \sim \dfrac{\hbar}{c \Sigma_y^{(\mathrm{II})} T_{\mathrm{II}} \Delta y^{(\mathrm{II})}} + \dfrac{c^2}{24} M_{\mathrm{I}} L_{\mathrm{I}}^2 T_{\mathrm{I}} \Delta \varepsilon_x^{(\mathrm{I})} | \bar{\tilde{B}}_{1122}^{(\mathrm{I},\mathrm{II})} - \bar{\tilde{B}}_{1122}^{(\mathrm{II},\mathrm{I})} | , \end{cases}
$$

from which the minimum value of their product is given by

$$\Delta\bar{E}_{11}^{(\mathrm{I})}\Delta\tilde{H}_{22}^{(\mathrm{II})}\sim\hbar\,|\,\tilde{\bar{B}}_{1122}^{(\mathrm{I},\mathrm{II})}-\bar{\tilde{B}}_{1122}^{(\mathrm{II},\mathrm{I})}\,|\,. \tag{10.19}$$

which is in complete harmony with the quantum gravitational formalism.

The measurabilities of E_{11} and E_{12} and of E_{11} and H_{12} can also be attained similarly as follows. Using a lever having two equal arms which make an angle of 45° with each other, springs having the same force constant and length are connected perpendicularly to the ends of the arms. The lever is again free to rotate about its joint. The other free ends of the springs are joined to the component particles P_{II} belonging to the second test body system and to a component particle P_{I} belonging to the first test body system. By this spring mechanism one may pursue a similar discussion and attain the results in agreement with the formalism.

11. – Measurability of two field averages with arbitrarily oriented axes.

Up to this point we have developed the measurement of the gravitational field strengths by considering two cubical test bodies of dimensions L whose three principal axes are parallel to each other. The principal axes of a cubical test body are defined as those which pass through the center of mass of the test body perpendicularly to its surfaces. We have limited our considerations on the measurements of the test body which is uniformly stretched in the x direction to the measurement of the E_{11} component. We have also limited our considerations on the measurement of the test body having the x component of spin density σ_x to the measurement of the H_{11} component.

We shall extend our treatment to the more general case in which it is not necessary for the principal axes to be parallel to one another, but in which the properties of the test body are kept the same. For this purpose it is convenient to consider merely the E_{11} and H_{11} components and to rotate one test body keeping the other fixed in the results of the previous treatment. The principal axes are assumed to serve the purpose of body co-ordinate axes for each test body and will be denoted by x_i^{I} and x_i^{II}. The direction cosines of the j-th axis of each test body may be represented by a_{ij} and b_{ij}. Hence

$$x_i = a_{ij}x_j^{\mathrm{I}}\,, \qquad x_i = b_{ij}x_j^{\mathrm{II}}\,. \tag{11.1}$$

In addition to the orthonormality conditions

$$a_{ij}a_{ik} = \delta_{jk}\,, \qquad b_{ij}b_{ik} = \delta_{jk}\,, \tag{11.2}$$

the cosine and sine of an angle between the j-th co-ordinate axis fixed in the first test body and the k-th axis fixed in the second test body may be expressed by means of the product of the a's and b's as

$$\cos\theta_{jk} = a_{ij}b_{ik}\,, \tag{11.3}$$

$$\lambda_p \sin\theta_{jk} = \varepsilon_{pil}a_{ij}b_{lk}\,, \tag{11.4}$$

where λ_p is a unit vector perpendicular to either axis. It is a well known fact that $\cos\theta_{jp}$, can be expressed by the Euler angles since we are concerned merely with the orientation of the body axes relative to each other.

It is desirable to start by considering the commutators between two gravitational field averages of the magnetic type. We properly rotate the body axes of one test body with respect to the space axes in which the second test body is fixed. According to (11.1) one finds from (3.3)

$$[\tilde{H}^{(\mathrm{I})'}_{ij}, \tilde{H}^{(\mathrm{II})'}_{p'q'}] = i\hbar(\tilde{K}^{(\mathrm{I},\mathrm{II})}_{ijp'q'} - \tilde{K}^{(\mathrm{II},\mathrm{I})}_{ijp'q'}) \tag{11.5}$$

with

$$K^{(\mathrm{I},\mathrm{II})}_{ijp'q'} = \frac{K}{4L_{\mathrm{I}}^3T_{\mathrm{I}}L_{\mathrm{II}}^3T_{\mathrm{II}}c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'(\mathscr{D}_{ip'}\mathscr{D}_{jq'} + \mathscr{D}_{iq'}\mathscr{D}_{jp'} - \mathscr{D}_{ij}\mathscr{D}_{p'q'})D^{\mathrm{ret}}(x'-x)\,, \tag{11.6}$$

where

$$\begin{cases} \mathscr{D}_{ip'} = \dfrac{\partial^2}{\partial x_i\partial x'_p} - \dfrac{\cos\theta_{ip'}}{c^2}\dfrac{\partial^2}{\partial t\partial t'}\,, \\ \mathscr{D}_{ij} = \dfrac{\partial^2}{\partial x_i\partial x_j} - \dfrac{\delta_{ij}}{c^2}\dfrac{\partial^2}{\partial t^2}\,, \end{cases} \tag{11.7}$$

and an analogous expression for $\mathscr{D}_{p'q'}$.

For the commutators between electric-type components we have

$$[\bar{E}^{(\mathrm{I})'}_{ij}, \bar{E}^{(\mathrm{II})'}_{p'q'}] = i\hbar[\bar{L}^{(\mathrm{I},\mathrm{II})}_{ijp'q'} - \bar{L}^{(\mathrm{II},\mathrm{I})}_{ijp'q'}] \tag{11.8}$$

with

$$\bar{L}^{(\mathrm{I},\mathrm{II})}_{ijp'q'} = \frac{9K}{L_{\mathrm{I}}^5T_{\mathrm{I}}L_{\mathrm{II}}^5T_{\mathrm{II}}c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'\,\omega(x,x')(\mathscr{D}_{ip'}\mathscr{D}_{jq'} + \mathscr{D}_{jp'}\mathscr{D}_{iq'} - \mathscr{D}_{ij}\mathscr{D}_{p'q'})\,, \tag{11.9}$$

where we have considered only weights in the x^{I} and x^{II} directions for each test body as mentioned at the beginning of this section.

The commutators between an electric-type component and a magnetic-type component can be obtained by a similar procedure. We rotate the body axes

of the first test body to arbitrary directions signified by i, j and k, (not parallel to the space-fixed body axes of the second test body) relative to the body axes of the other, which are signified by p', q' and r'. By virtue of (3.2) and by the same method as used in deriving the commutators between two magnetic-type components, we get

$$[\bar{E}_{ij}^{(\mathrm{I})'}, \tilde{H}_{p'q'}^{(\mathrm{II})'}] = i\hbar(\tilde{\bar{G}}_{ijp'q'}^{(\mathrm{I,II})} - \bar{\tilde{G}}_{ijp'q'}^{(\mathrm{II,I})}) \tag{11.10}$$

with

$$\tilde{\bar{G}}_{ijp'q'}^{(\mathrm{I,II})} = \frac{3K}{2L_{\mathrm{I}}^5 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x' \, \omega(x) \cdot \tag{11.11}$$

$$\cdot \left\{ (\mathscr{D}_{ip} \cos\theta_{q'n} \varepsilon_{jnl} + \mathscr{D}_{jp'} \cos\theta_{q'n} \varepsilon_{inl}) \frac{\partial}{\partial x_l} - \mathscr{D}_{ij} \cos\theta_{r'n} \varepsilon_{p'q'r'} \frac{\partial}{\partial x_n} \right\} \frac{\partial}{c\,\partial t} D^{\mathrm{ret}}(x' - x) \, .$$

Here the expression is obtained by taking the weight in the x^{I} direction in the body co-ordinates fixed to the first body, for which we measure the electric-type field component.

Following the procedure used in deriving (6.11) we get (*)

$$\Delta\bar{E}_{p'q'}^{(\mathrm{I,II})'} = \frac{3KM\,\Delta\varepsilon_x^{(\mathrm{I})}}{4L_{\mathrm{I}}^3 L_{\mathrm{II}}^5 T_{\mathrm{II}} c} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x' \, \omega(x, x') \cdot \tag{11.12}$$

$$\cdot (\mathscr{D}_{1p'} \mathscr{D}_{1q'} + \mathscr{D}_{1q'} \mathscr{D}_{1p'} - \mathscr{D}_{11} \mathscr{D}_{p'q'}) \nabla^4 D^{\mathrm{ret}}(x' - x) \, .$$

Similarly for the magnetic-type components we get

$$\Delta\tilde{H}_{p'q'}^{(\mathrm{I,II})'} = \frac{KM\,\Delta\varepsilon_x^{(\mathrm{I})}}{8L_{\mathrm{I}}^3 T_{\mathrm{I}} L_{\mathrm{II}}^3 T_{\mathrm{II}} c} \int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x' \, \omega(x) \cdot \tag{11.13}$$

$$\cdot \left\{ (\mathscr{D}_{1p'} \cos\theta_{qn} \varepsilon_{1nl} + \mathscr{D}_{1p} \cos\theta_{qn} \varepsilon_{1nl}) \frac{\partial}{\partial x_l} - \mathscr{D}_{11} \cos\theta_{r'n} \varepsilon_{p'q'r'} \frac{\partial}{\partial x_n} \right\} \frac{\partial}{c\,\partial t'} D^{\mathrm{ret}}(x' - x) \, .$$

In eq. (11.12), (11.13) we have considered only the field uncertainties in the region II resulting from the uncertainty of $\varepsilon_x^{(\mathrm{I})}$ in the region I.

Putting $i = j = 1$ in eq. (11.8) and (11.10) and making use of the Un-

(*) See Appendix II.

certainty Principle, we obtain

$$(11.14)\quad \Delta\bar{E}_{11}^{(\mathrm{I})'}\Delta\bar{E}_{p'q'}^{(\mathrm{II})'}\sim\frac{i\hbar\cdot 9K}{L_{\mathrm{I}}^5T_{\mathrm{I}}L_{\mathrm{II}}^5T_{\mathrm{II}}c}\int_{\mathrm{II}}\mathrm{d}^4x\int_{\mathrm{I}}\mathrm{d}^4x'\,\omega(x,x')(2\mathscr{D}_{1p'}\mathscr{D}_{1q'}-\mathscr{D}_{11}\mathscr{D}_{p'q'})\,D^{\mathrm{ret}}(x'-x),$$

$$(11.15)\quad \Delta\bar{E}_{11}^{(\mathrm{I})'}\Delta\tilde{H}_{p'q'}^{(\mathrm{II})'}\sim\frac{3i\hbar}{2L_{\mathrm{I}}^5T_{\mathrm{I}}L_{\mathrm{II}}^3T_{\mathrm{II}}c}\int_{\mathrm{I}}\mathrm{d}^4x\int_{\mathrm{II}}\mathrm{d}^4x'\,\omega(x)\cdot$$

$$\cdot\left(2\mathscr{D}_{1p'}\cos\theta_{qn}\varepsilon_{1nl}\frac{\partial}{\partial x_l}-\mathscr{D}_{11}\cos\theta_{r'n'}\varepsilon_{p'q'r'}\frac{\partial}{\partial x_n}\right)\frac{\partial}{c\,\partial t'}D^{\mathrm{ret}}(x'-x)\,.$$

Comparing eq. (11.12) and (12.13) with (11.14) and (11.15) respectively, we find

$$(11.16)\quad \Delta\bar{E}^{(\mathrm{I})'}\sim\frac{12\hbar}{c^2M_{\mathrm{I}}L_{\mathrm{I}}^2T_{\mathrm{I}}\Delta\varepsilon_x^{(\mathrm{I})}}\,.$$

Hence we have arrived at the conclusion that one is required to measure only the weighted average value of the E_{11} component.

A similar treatment can also be carried out for magnetic-type components on the measurement of two cubical test bodies whose principal axes are not parallel to one another. Therefore we reach an analogous conclusion, namely, that one needs only to measure the $\tilde{H}_{11}$ component. The measurability of two test bodies whose principal axes are not parallel to each other can also be attained by an argument similar to Sections **9**, **10** and **11** if we replace the lever with two perpendicular arms by a lever with two arms which have a properly replaced angle between them.

12. – Conclusions.

We have shown that the limitations determined by physically possible measurements are in complete agreement with those imposed on the measurability of the quantized gravitational field strength. It has also been shown that in order to measure the field strengths one is required to take a field average over a space-time region. For magnetic-type components the measurement procedure and field averages are similar to those of the electric magnetic field strengths while for electric-type components field averages are hyperbolic weighted averages and for the measurement of electric-type field averages

it is necessary to measure the strain of an elastic medium which serves as a test body. The disturbances of the field averages in one spatial region caused by the measurement of some field components in the other can be obtained from Einstein's equation. These disturbances completely agree with the results from the quantum gravitational formalism taking into account the dynamical equations of motion of the test bodies together with the limitations imposed on the measured quantities (the strains and strain momenta for the electric-type component and the co-ordinates and momenta for the magnetic-type components) by the Uncertainty Principle. The self-reactions of the test bodies during a measurement-time interval can be compensated for by introducing appropriate spring mechanisms as Bohr and Rosenfeld did in the measurement of the electromagnetic field strengths.

* * *

The author wishes to express his profound gratitude to Dr. Bryce S. DeWitt under whose inspiring guidance this work was carried out. Thanks are also due to Drs. B. S. DeWitt and C. M. DeWitt for the financial support and the members of Physics Department, University of North Carolina for the hospitality extended to the author during his stay at the Department.

Appendix I

In the quantum electromagnetic formalism if there is a uniform polarization $\varrho\Delta x$ per unit volume in a spatial region I the uncertainties of field strengths at (x_2, y_2, z_2, t_2) are given by

$$\Delta F_{\mu\nu}(x_2) = \int [(\Delta j_\nu)_{,\mu} - (\Delta j_\mu)_{,\nu}] D^{\mathrm{ret}}(x_2 - x_1)\, \mathrm{d}^4 x_1 \,. \tag{A.I.1}$$

Without finding four potentials as derived in reference [2], we can easily compute these uncertainties by means of the method described in Sect. 6 by using

$$\Delta j_4 = -\frac{\partial P}{\partial x}\,, \qquad \Delta j_1 = c\,\frac{\partial P}{\partial x_0}\,, \qquad \Delta j_2 = \Delta j_3 = 0\,, \tag{A.I.2}$$

where

$$P = \varrho\,\Delta x\,\theta\left(x + \frac{L}{2}\right)\theta\left(\frac{L}{2} - x\right)\theta\left(y + \frac{L}{2}\right)\theta\left(\frac{L}{2} - y\right)\theta\left(z + \frac{L}{2}\right)\theta\left(\frac{L}{2} - z\right)\theta(t - t')\,\theta(t'' - t).$$

Appendix II

According to eq. (6.7) we have

$$\text{(A.II.1)}\qquad \Delta E_{pq}(x') = \frac{K}{2}\int \mathrm{d}^4x \left[\left(\Delta T_{00} + \frac{\Delta T}{2}\right)\frac{\partial^2}{\partial x'_p \partial x'_q} - 2\Delta T_{p0}\frac{\partial^2}{c\,\partial t' \partial x'_q} + \right.$$
$$\left. + \left(\Delta T_{pq} - \frac{\delta_{pq}}{2}\Delta T\right)\frac{\partial^2}{c^2 \partial t^2}\right] D^{\mathrm{ret}}(x'-x)\,.$$

By rotating the three principal axes of the second test body through arbitrary angles with respect to the space co-ordinate in which the first test body is fixed, three direction cosines of the new i-th body axes are given by b_{pi}. Multiplying the above equation by $b_{p1}b_{q1}$, summing over p and q and denoting $\Delta E_{pq}b_{p1}b_{q1}$ by $\Delta E'_{11}$, we obtain

$$\text{(A.II.2)}\qquad \Delta E'_{11}(x') = \frac{K}{4}\int \mathrm{d}^4x \left[(2\Delta T_{00} + \Delta T))\frac{\partial^2}{\partial x'^2} - 2\Delta T'_{10}\frac{\partial^2}{c\,\partial t' \partial x'_p} + \right.$$
$$\left. + (2\Delta T'_{11} - \Delta T)\frac{\partial^2}{c^2 \partial t'^2}\right] D^{\mathrm{ret}}(x'-x)\,.$$

Substituting $\Delta T'_{10} = b_{i1}T_{i0}$, $\Delta T'_{11} = b_{i1}b_{j1}T_{ij}$ and relations (6.6) into (A.II.2), we get

$$\text{(A.II.3)}\qquad \Delta E'_{11}(x') = \frac{K}{4}\int \mathrm{d}^4x \left[\left(\frac{\partial^2 P_e}{\partial x^2} + \frac{\partial^2 P_e}{c^2 \partial t^2}\right)\frac{\partial^2}{\partial x'^2} - 4\cos\theta_{11'}\frac{\partial^2 P_e}{c^2 \partial x\,\partial t}\frac{\partial^2}{\partial x'\,\partial t'} + \right.$$
$$\left. + 2\cos^2\theta_{11'}\frac{\partial^2 P_e}{c^4 \partial t^2}\frac{\partial^2}{\partial t'^2} - \left(\frac{\partial^2 P_e}{\partial x^2} - \frac{\partial^2 P_e}{c^2 \partial t^2}\right)\frac{\partial^2}{c^2 \partial t'^2}\right] D^{\mathrm{ret}}(x'-x)\,.$$

Referring to the derivation of (6.10) we can write

$$\text{(A.II.4)}\qquad \Delta E'_{11}(x') = \frac{KM\Delta\varepsilon_x^{(\mathrm{I})}}{4\bar{L}_{\mathrm{I}}^3}\int_{\mathrm{I}} \mathrm{d}^4x\,\omega(x)(2\mathscr{D}_{11'}\mathscr{D}_{11'} - \mathscr{D}_{11}\mathscr{D}_{1'1'})\,D^{\mathrm{ret}}(x'-x)\,,$$

from which, by taking the weighted average over the region II, we have

$$\text{(A.II.5)}\qquad \Delta E_{11}^{(\mathrm{I,II})} = \frac{3KM\Delta\varepsilon_x^{(\mathrm{I})}}{2\bar{L}_{\mathrm{I}}^3 L_{\mathrm{II}}^5 T_{\mathrm{II}} c}\int_{\mathrm{I}} \mathrm{d}^4x \int_{\mathrm{II}} \mathrm{d}^4x'\,\omega(x,x')(2\mathscr{D}_{11'}\mathscr{D}_{11'} - \mathscr{D}_{11}\mathscr{D}_{1'1'})\,D^{\mathrm{ret}}(x'-x).$$

Generally speaking, $\Delta E_{pq}^{(\mathrm{I,II})'}$ reduces to (11.12). In a similar fashion one can easily compute the magnetic-type components induced by an uncertainty in an electric-type component in the region II, and the field uncertainties as well.

SUPPLEMENTO AL VOLUME XXI, SERIE X
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Selected Topics on Analyticity in Potential Scattering.

A. MARTIN

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(ricevuto il 22 Luglio 1961)

These lectures were prepared for the summer school of Theoretical Physics in Yugoslavia (1961). They are essentially based on work done by the CERN group, the Torino group and the Padua group. Though, in the text, reference is made to other lecturers, these lectures can be considered as self contained. They do not cover all the subject but the interested reader will easily find in the references what is missing.

1. – Introduction.

In recent years a large amount of work has been done on the analytic properties of scattering and production amplitudes in field theory. However, at the time where these studies were started, very little had been done about the analytic properties in potential scattering except in the case where the interaction took place in a strictly finite region [1]. The first really important

[1] E. P. WIGNER: *Phys. Rev.*, **98**, 145 (1955).

contribution in this field was that of KHURI [2] who showed that the scattering amplitude derived from a Yukawa-like potential obeys a fixed transfer dispersion relation.

More recently, assumptions have been made about the analytic properties of the scattering amplitude in field theory, both in energy and momentum transfer [3]. There does not exist yet any complete proof that these assumptions (Mandelstam representation) are valid. However, in this case, the potential model is a useful guide which shows that at least for generalized Yukawa potential the Mandelstam representation holds [4].

We shall not speak here about the complete derivation of this result which has been gyen by Prof. T. REGGE here. We shall first consider the analytic properties of a given partial wave. These analytic properties would be obtained by projecting the Mandelstam representation on a given angular momentum state. However, a direct method of obtaining the analytic properties of a partial wave proves very useful. Indeed, it turns out that the method we shall follow for Yukawa-like potentials furnishes many other results. In particular it furnishes a very nice integral representation of the wave functions (especially of the deuteron wave function) which permits to study certain matrix elements such as deuteron electromagnetic form factors and deuteron photodisintegration and to show that the so-called anomalous thresholds (at least the « normal » anomalous thresholds) are connected with the extension of the deuteron wave function.

In addition, the discontinuities of the l^{th}-wave scattering amplitude across the unphysical cut can be expressed explicitly from the knowledge of the potential. It turns out that the reverse is true: given the discontinuity one can construct explicitly the potential. Since the discontinuities are at present the quantities which the dispersionists calculate first one sees that for a given partial wave one can associate a potential and, therefore a wave function. In this way one can connect the information about the discontinuities for nucleon-nucleon scattering and the external part of the deuteron wave function.

A similar connection between potential and discontinuity can be established in the case of fixed energy dispersion relations. We shall have to sketch first a method to establish the analytic properties for fixed t which is completely different from what you heard from Dr. REGGE. Then we shall be able to

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[2] N. N. KHURI: *Phys. Rev.*, **107**, 1148 (1957); see also: A. KLEIN and C. ZEMACH: *Ann. Phys.*, **7**, 440 (1959).

[3] S. MANDELSTAM: *Phys. Rev.*, **112**, 1344 (1958).

[4] A. KLEIN: *Journ. Math. Phys.*, **1**, 41 (1960); R. BLANCKENBECKLER, M. GOLDBERGER, N. KHURI and S. TREIMAN: *Ann. Phys.*, **10**, 62 (1960); T. REGGE: *Nuovo Cimento*, **14**, 951 (1959); J. BOWCOCK and A. MARTIN: *Nuovo Cimento*, **14**, 516 (1959).

show how one can construct from it the potential. As an application we shall show how one can settle the problem of the range of the imaginary potential in nucleon-antinucleon scattering.

2. – Analytic properties of partial waves for Yukawa-like potentials.

2·1. *S wave case* (5). – In the preceding lectures Prof. T. REGGE has shown to you the analytic properties of finite range potentials and, also, established the existence of an analyticity strip for potentials decreasing faster than an exponential. We shall consider here the analytic properties for potentials of the type

$$V(r) = \int_{\mu}^{\infty} C(\alpha) \exp[-\alpha r] \, \mathrm{d}\alpha \,, \tag{1}$$

which include the Yukawa potential ($C(\alpha) = \text{const}$).

It proves useful to consider the following two solutions of the Schrödinger equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + k^2 - V(r)\right] u^{\pm}(k, r) = 0 \,, \tag{2}$$

$$\begin{cases} u^{+}(k, r) = \exp[ikr] \, f(-k, r) \,, & f(-k, \infty) = 1 \,, \\ \text{with} \\ u^{-}(k, r) = \exp[-ikr] f(k, r) \,, & f(k, \infty) = 1 \,, \end{cases} \tag{3}$$

whether or not such solutions exist for complex values of k will be established by trying to construct f. Notice that $f(k, 0)$ is just the Jost function about which you have already heard.

If the function $f(k, r)$ is known, one can easily construct the S matrix (I should say S functions), defined by

$$u_{\text{regular}} \sim \exp[-ikr] - S(k) \exp[ikr] \,,$$

(for large r) at least for real k.

Since

$$u_{\text{regular}} = \frac{u^{+}(k, r)}{u^{+}(k, 0)} - \frac{u^{-}(k, r)}{u^{-}(k, 0)} \,,$$

(5) A. MARTIN: *Nuovo Cimento*, **14**, 403 (1959).

we see that

$$S(k) = \frac{f(k, 0)}{f(-k, 0)} . \tag{4}$$

For complex values of k, the latter expression will be maintained as a definition of $S(k)$.

Let us now speak about the construction of f.

In the especially simple case of a pure exponential potential [6] $(C(\alpha) = \delta(\alpha - \mu))$ the solution of the S wave Schrödinger equation is known and one can easily recognize that $f(k, r)$ can be expanded as a series of exponentials of the type

$$f(k, r) = 1 + \sum_{n=1}^{\infty} C_n(k) \exp[-n\mu r] ,$$

for a potential $V = V_0 \exp[-\mu r]$.

The same can be done for more complicated potentials

$$V = \sum_{N=1}^{M} V_N \exp[-\mu_N r] ,$$

for which no exact solution is known. Here one can write

$$f(k, r) = 1 + \sum C_{n_1 n_2 \dots n_M} \exp[-(n_1\mu_1 + \mu_2 n_2 \dots + n_M \mu_M)]r .$$

This strongly suggests that when $V(r)$ is a continuous superposition of exponentials in which the largest range is $1/\mu$, $f(k, r)$ can be written as

$$f(k, r) = 1 + \int_{\mu}^{\infty} \varrho_k(\alpha) \exp[-\alpha r] \, d\alpha . \tag{5}$$

We shall take this formula as a starting point and check later on that our assumption was correct.

Inserting $u^- = f(k, r) \exp[-ikr]$ in the Schrödinger equation we get

$$f'' - 2ikf' - V(r)f = 0 . \tag{6}$$

Replacing V and f by their integral representations (1) and (5) respectively, and using the well-known theorem about the product of two Laplace trans-

[6] R. Jost: *Helv. Phys. Acta*, **9**, 256 (1946).

forms (which appear in Vf), we end up with an equation for $\varrho_k(\alpha)$:

$$\alpha(\alpha + 2ik)\varrho_k(\alpha) = C(\alpha) + \int C(\alpha - \beta)\varrho_k(\beta)\,\mathrm{d}\beta\,. \tag{7}$$

Now we have to fix the upper and lower limits in the integral. Obviously the lower limit is μ because $C(\alpha) = 0$ for $\alpha < \mu$. From this fact we easily see that $\varrho_k(\alpha) = 0$ in the range $\alpha = 0$, $\alpha = \mu$, which is consistent with the assumption we made. Finally we can write

$$\alpha(\alpha + 2_i k)\varrho_k(\alpha) = C(\alpha) + \theta(\alpha - 2\mu)\int_{\mu}^{\alpha-\mu} C(\alpha - \beta)\varrho_k(\beta)\,\mathrm{d}\beta\,. \tag{8}$$

This integral equation has the great advantage that it can be solved very easily by iteration for any finite value of α:

$$\text{for} \quad 0 < \alpha < \mu \qquad \varrho = 0\,,$$

$$\mu < \alpha < 2\mu \qquad \varrho = \frac{C(\alpha)}{\alpha(\alpha + 2ik)}\,,$$

$$2\mu < \alpha < 3\mu \qquad \varrho = \frac{1}{\alpha(\alpha + 2ik)}\left[C(\alpha) + \int_{\mu}^{\alpha-\mu} \frac{C(\alpha - \beta)\,C(\beta)}{\beta(\beta + 2ik)}\,\mathrm{d}\beta\right],$$

and, more generally, if $\varrho_k(\alpha)$ is known for $\alpha < n\mu$ one can insert it in the right-hand side and obtain ϱ for $\alpha < (n+1)\mu$.

It is clear that if $(\alpha + 2ik)$ cannot vanish, *i.e.* if k lies outside a line on the imaginary axis for $k = i(\mu/2)$ to $i\infty$, $\varrho_k(\alpha)$, for fixed α, is a well defined function of k, is analytic in k. We have now to see what kind of condition should be imposed on $C(\alpha)$ to ensure that these properties hold for $f(k, r)$. We shall look for *sufficient* but not necessary conditions.

Let us assume that there exist a monotonous increasing function $B(\alpha)$ such that

$$|C(\alpha)| < B(\alpha)\,. \tag{9}$$

Let us take k outside the cut on the imaginary axis; more precisely we exclude the domain indicated on Fig. 1, where ε is an arbitrarily angle then, for $\alpha > \mu$

$$|\alpha + 2ik| > \alpha \sin\varepsilon\,,$$

$$|\alpha + 2ik| > 2|k| \sin\varepsilon\,,$$

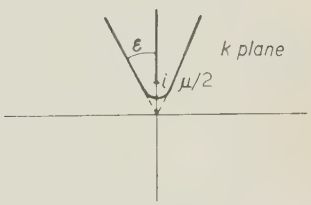


Fig. 1.

then we get

$$|\varrho_k(\alpha)| < \frac{B(\alpha)}{\alpha^2 \sin \varepsilon}\left[1 + \int_{\mu}^{\alpha} |\varrho_k(\beta)|\,\mathrm{d}\beta\right]. \tag{10}$$

Let us call

$$1 + \int_{\mu}^{\alpha} |\varrho_k(\beta)|\,\mathrm{d}\beta = y_k(\beta)\,,$$

then

$$\frac{y'}{y} < \frac{B(\alpha)}{\alpha^2 \sin \varepsilon}\,.$$

Hence

$$y_k(\alpha) < \exp\left[\int_{\mu}^{\alpha} \frac{\mathrm{d}\beta B(\beta)}{\beta^2 \sin \varepsilon}\right].$$

It is clear now that if $\int_{\mu}^{\infty} \mathrm{d}\beta B(\beta)/\beta^2$ is bounded $1 + \int_{\mu}^{\infty} |\varrho_k(\alpha)|\,\mathrm{d}\alpha$ is uniformly bounded with respect to k when k is outside the neighbourhood of the cut. Therefore a sufficient condition for the existence and holomorphy of $f(k, r) = 1 + \int_{\mu}^{\infty} \exp[-\alpha r]\varrho_k(\alpha)\,\mathrm{d}\alpha$, for fixed arbitrary r with respect to k is

$$|C(\alpha)| < M\alpha^{1-\eta}, \qquad \eta \text{ arbitrarily small:} \tag{11}$$

another sufficient condition would be

$$|C(\alpha)| < \frac{M\alpha}{\lambda + \ln^{1+\eta}(\alpha/\mu)}\,. \tag{12}$$

This ensures, in particular that $f(k, 0)$ is regular outside the cut $i(\mu/2) - i\,\infty$. Further, making use of the uniform convergence property we see easily that since $\varrho_k(\alpha) \to 0$ as $k \to \infty$ in any direction except the positive imaginary axis

$$\lim_{|k| \to \infty} f(k, r) = 1\,. \tag{13}$$

Further, in the case when condition (11) is fulfilled a more careful examination

shows that

$$(14)\qquad \begin{cases} |f(k,r)-1)| \sim |k|^{-\eta} & \text{for} \quad \eta<1\,, \\ \text{as } |k|\to\infty \quad \sim \dfrac{\ln|k|}{|k|} & \text{for} \quad \eta\geqslant 1\,. \end{cases}$$

The conditions imposed on $C(\alpha)$ essentially mean that $V(r)$ has to be less singular than $1/r^2$ at the origin of coordinates. We know that if this is not so difficulties arise, and in particular for sufficiently strong $1/r^2$ attractive potentials the Schrödinger equation has no regular solution. The question whether one can deal with very singular repulsive potentials using our method is not yet settled, but we would like to show that when $C(\alpha)$ is bounded by an arbitrary polynomial $f(k,r)$ retains all the desired properties for $r\neq 0$. In this case

$$(15)\qquad |C(\alpha)\exp[-\alpha r]|,$$

is bounded by, say, B; then

$$(16)\qquad |\varrho_k(\alpha)\exp[-\alpha r]|<\frac{B}{\alpha^2\sin\varepsilon}\left[1+\int_{\mu}^{\alpha}|\varrho_k(\beta)\exp[-\beta r]|\,\mathrm{d}\beta\right];$$

hence one shows easily that

$$1+\int_{\mu}^{\infty}|\varrho_k(\alpha)\exp[-\alpha r]|\,\mathrm{d}\alpha\,,$$

is uniformly convergent and all the above mentioned properties hold.

Up to now we have established the analytic properties of the outgoing and ingoing wave functions for fixed r. In case condition (11) or (12) are satisfied it is an almost trivial matter to get the analytic properties of

$$S(k)=\frac{f(k,0)}{f(-k,0)}\,.$$

It has two cuts from $k=i\mu/2$ to $k=i\infty$ and from $k=-i\infty$ to $k=-i\mu/2$. To show that these are real cuts we should establish explicitly the existence of a discontinuity of $S(k)$ across these lines. This will be postponed to Sect. 3. In addition to these cuts $S(k)$ has poles which come from the zeros of $f(-k,0)$. You have probably already heard from Regge that the zeros of $f(-k,0)$ in the upper half k plane necessarily lie on the imaginary axis (*) and correspond

(*) This is due to the reality of the potential. If $f(-k,0)=0$, with $\mathrm{Im}\,k>0$, $u^+(k,r)$ is regular at the origin and vanishes at infinity. Hence by combination of $(\mathrm{d}^2/\mathrm{d}r^2+k^2)u=Vu$ and $(\mathrm{d}^2/\mathrm{d}r^2+k^{*2})u^*=Vu^*$ one gets $(k^2-k^{*2})\int u^*u\,\mathrm{d}r=0$.

to bound states whose wave function is just

$$\exp[-\varkappa r]\left[1+\int_{\mu}^{\infty}\varrho_{i\varkappa}(\alpha)\exp[-\alpha r]\,\mathrm{d}\alpha\right], \qquad \varkappa \text{ real} > 0\,. \tag{17}$$

From $f(k, 0) \to 1$ for $k \to \infty$ we get $S(k) \to 1$. Using the asymptotic behaviour of $f(k, 0) - 1$ we can show that, provided condition (11) is fulfilled, one can write *unsubtracted* Cauchy integral representations (or dispersion relations) for

$$\frac{S(k)-1}{2i} = \exp[i\delta]\sin\delta\,.$$

Finally, taking into account the symmetries of f:

$$f(k, 0) = f^*(-k^*, 0)\,, \tag{18}$$

we get the following symmetries for S

$$S(k)S(-k) = 1\,, \qquad S(k) = S^*(-k^*)\,, \tag{19}$$

and the results are summarized in fig 2.

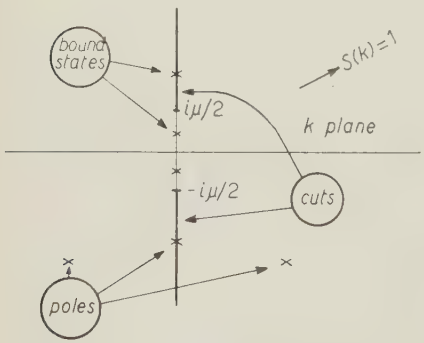


Fig. 2.

As a last comment of this section we would like to say that when $C(\alpha)$ increases like an arbitrary polynomial $f(k, 0)$ and $f(-k, 0)$ do not exist. However, we believe (without being able to prove it at present) that

$$S(k) = \lim_{r\to 0}\frac{f(k, r)}{f(-k, r)}\,,$$

might still exist if $C(\alpha)$ is positive (repulsive potential at short distances) for large α. The analytic properties would remain unchanged except perhaps the behaviour at infinity. Our belief is based on the remark that for a hard core potential:

$$\begin{cases} V(r) = +\infty\,, & \text{for} \quad r < r_0\,, \\ V(r) = \displaystyle\int_{\mu}^{\infty} C(\alpha)\exp[-\alpha r]\,\mathrm{d}\alpha & \text{for} \quad r > r_0\,, \end{cases} \tag{20}$$

where $C(\alpha)$ is bound by a polynomial the analytic properties still hold, but, at infinity, $S(k) \simeq \exp[2ikr_0]$. Then, we believe that letting r_0 go to zero will not seriously change the picture, or, in other words, since here we have replaced the singular region by a worse singularity in the potential, nothing worse than what happened in the hard core case can happen.

2·2. *Higher angular momenta* (7). – Here all the trouble comes from the centrifugal terms in the Schrödinger equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{l(l+1)}{r^2} + k^2 - V(r)\right] u^{\pm}(k, r) = 0 , \tag{21}$$

In this case we have two possibilities:

— either to keep that same integral representation

$$u^{-}(k, r) = \exp[-ikr]\left|1 + \int_{\mu}^{\infty} \varrho_k(\alpha) \exp[-\alpha r]\,\mathrm{d}\alpha\right| . \tag{22}$$

but then, since the range of the centrifugal potential is infinite we expect that the lower limit in the integral representation will be zero rather than μ and the iterative property of the integral equation for ϱ is lost at least in appearance.

— or to try to generalize the S wave case by using functions which satisfy the free equations (with centrifugal term included). In the S wave case we had

$$u^{-}(k, r) = \exp[-ikr] + \int_{\mu}^{\infty} \varrho_k(\alpha) \exp[-(ik+\alpha)r]\,\mathrm{d}\alpha ,$$

in the $l \neq 0$ case one can take

$$u^{-}(k, r) = i^{l+1} kr \left|h_l^{(2)}(kr) + \int_{\mu}^{\infty} \varrho_k(\alpha) h_l^{(2)}[(k+i\alpha)r]\,\mathrm{d}\alpha\right| , \tag{23}$$

where $h_l^{(2)}$ is defined as in SCHIFF: *Quantum Mechanics*, p. 79.

The latter method which was used by DE ALFARO and ROSSETTI (8) avoids the difficulties associated with the centrifugal term itself but, one has to be

(7) A. MARTIN: *Nuovo Cimento*, **15**, 99 (1959).
(8) V. DE ALFARO and M. ROSSETTI: *Nuovo Cimento*, **18**, 783 (1960).

able to write the product $u\ (k, r)V(r)$ as a superposition of Hankel functions, which is possible but rather tedious and difficult.

Here we shall prefer to use the first method and show how one can get rid of the centrifugal term. Let us first merely include the centrifugal term in the potential. Since

$$\frac{1}{r^2} = \int_0^\infty \alpha \exp[-\alpha r]\,\mathrm{d}\alpha\,,$$

it is not difficult to see that starting from

$$f(k, r) = 1 + \int_0^\infty \varrho_k(\alpha) \exp[-\alpha r]\,\mathrm{d}\alpha\,,$$

one gets

$$(24)\quad \alpha(\alpha + 2ik)\varrho_k(\alpha) = C(\alpha) + l(l+1)\alpha + \int_0^\lambda [C(\alpha - \beta) + l(l+1)(\alpha - \beta)]\varrho_k(\beta)\,\mathrm{d}\beta\,.$$

What one would like to do is to transform this equation into an iterative type integral equation. One can notice that when $C(\alpha) = 0$ the solution of this equation is known, because it can be obtained by means of Laplace transformation from the Hankel functions. We shall prefer to show it directly. Let us call $\varrho_k^0(\alpha)$ the solution for $C(\alpha) \equiv 0$

$$(25)\qquad \alpha(\alpha + 2ik)\varrho_k^1(\alpha) = l(l+1)\left[\alpha + \int_0^\lambda (\alpha - \beta)\varrho_k^0(\beta)\,\mathrm{d}\beta\right],$$

by differentiation of both sides of this equation we readily get a differential equation for the quantity

$$(26)\qquad \begin{cases} y_k^0(\alpha) = 1 + \int_0^\lambda \varrho_k^0(\beta)\,\mathrm{d}\beta, \\ [\alpha(\alpha + 2ik)y_k^{0\prime}(\alpha)]' = l(l+1)y_k^0(x)\,: \end{cases}$$

if we compare this with the Legendre equation

$$(27)\qquad -[(1 - x^2)u']' = l(l+1)u\,,$$

we readily see that $P_l(1+(\alpha/ik))$ and $Q_l(1+(\alpha/ik))$ are solutions of the equation for $y_k^0(\alpha)$.

Actually, since $\varrho_k^0(\alpha)$ is regular at $\alpha = 0$, and since $y_k^0(0) = 1$, we conclude that

$$\varrho_k^0(\alpha) = \frac{\mathrm{d}P_l(1+(\alpha/ik))}{\mathrm{d}\alpha} . \tag{28}$$

Let us notice that $P(1+(\alpha/ik))$ is regular in the whole complex k plane, with a pole at $k = 0$, which means that even for $k = ix$ the equation for ϱ_k^0 has a well behaved solution. It will be useful to notice that the other solution $Q_n(1+(\alpha/ik))$, for fixed α, has — and it can be shown from

$$Q_n(z) = \frac{1}{2}\int_{-1}^{+1} \frac{P_n(t)\,\mathrm{d}t}{z-t} ,$$

a cut along $k = iX$ where X is a real number going from $\frac{1}{2}$ to $+\infty$.

Let us now return to the equation for $\varrho(\alpha)$ in the case $C(\alpha) \neq 0$. It may be written

$$\alpha(\alpha+2ik)\varrho_k(\alpha) - l(l+1)\left[\alpha + \int_0^\alpha (\alpha-\beta)\,\varrho_k(\beta)\,\mathrm{d}\beta\right] = \tag{29}$$

$$= C(\alpha) + \int_0^\alpha C(\alpha-\beta)\,\varrho_k(\beta)\,\mathrm{d}\beta = \varphi(\alpha) .$$

Notice that $\varphi(\alpha) = 0$ for $\alpha < \mu$; so for $\alpha < \mu$, $\varrho_k(\alpha) \equiv \varrho_k^0(\alpha)$. We can as before use $y_k(\alpha) = 1 + \int_0^\alpha \varrho_k(\beta)\,\mathrm{d}\beta$ and get

$$[\alpha(\alpha+2ik)y_k'(\alpha)]' = l(l+1)\,y_k(\alpha) = \frac{\mathrm{d}\varphi}{\mathrm{d}\alpha} . \tag{30}$$

φ will be treated as a known right-hand side according to standard methods. Then one can formally solve the equation and obtain

$$y_k(\alpha) = P_l\left(1+\frac{\alpha}{ik}\right) \tag{31}$$

$$+\int_0^\alpha \left| P_l\left(1+\frac{\alpha}{ik}\right) Q_l\left(1+\frac{\beta}{ik}\right) - P_l\left(1+\frac{\beta}{ik}\right) Q_l\left(1+\frac{\alpha}{ik}\right)\right| \frac{\mathrm{d}\varphi}{\mathrm{d}\beta}\,\mathrm{d}\beta .$$

In this solution a choice of the homogeneous solution has been made: it is such that for $\alpha < \mu$, $y_k(\alpha) \equiv y_k^0(\alpha)$ since for $\alpha < \mu$ $\mathrm{d}\varphi/\mathrm{d}\alpha = 0$.

At this point we could still transform a little bit the equation, but the main point is already exhibited here:

$$\varphi(\alpha) = C(\alpha) + \int_0^{\alpha-\mu} C(\alpha-\beta) y_k'(\beta)\,\mathrm{d}\beta\,,$$

(because $C(\alpha)=0$ for $\alpha<\mu$).

Therefore, if we know $y(\beta)$ for $\beta<n\mu$ we know $\varphi(\beta)$ for $\beta<(n+1)\mu$ and inserting this in the r.h.s. of eq. (31) we obtain $y_k(\beta)$ for $\beta<(n+1)\mu$. In this new form the iterative property is re-established.

One might think that a troublesome underlying assumption of our method of transformation of the equation is the derivability of $q(\alpha)$ especially at the point $\alpha=\mu$ (which is not valid for a pure Yukawa potential). Actually, by partial integration and derivation one can transform the equation into an equation for $\varrho_k(\alpha)$ where $q(\beta)$ rather than $\mathrm{d}q/\mathrm{d}\beta$ appears. It is not difficult to believe that the lack of derivability of $q(\beta)$ will not alter the final result. We shall not, however, enter into these details. From what we know about the analytic properties of Q and P it is clear that $\varrho_k(\alpha)$, again, is analytic provided k is outside the origin and outside the cut $k=i\mu/2\to i\,\infty$. To prove that this is also true for $f(k,r)$ is tedious, because one needs to introduce upper bounds of the various functions appearing in the equation, but nevertheless possible. All the results which have been established in the S wave case can be proved in the present case. To establish the properties of $S(k)=\lim_{r\to 0} f(+k,r)/f(-k,r)$, one can notice that $S(k)$ is also $\lim_{\alpha\to\infty} \varrho_k(\alpha)/\varrho_{-k}(\alpha)$.

2·3. *Case where tensor forces are present* (9). – Since the actual nucleon-nucleon forces are not central but contain spin orbit and tensor forces, it is desirable to extend our method to the case where $L=J-1$ and $L=J+1$ states are coupled through a non-central potential. Then the starting equations are

$$(32)\qquad \begin{cases} \left[\dfrac{\mathrm{d}^2}{\mathrm{d}r^2} + k^2 - \dfrac{J(J-1)}{r^2} - V_{J-1}(r)\right] u_{J-1}(r) = V_T(r)\, u_{J+1}(r)\,, \\ \left[\dfrac{\mathrm{d}^2}{\mathrm{d}r^2} + k^2 - \dfrac{(J+1)(J+2)}{r^2} - V_{J+1}(r)\right] u_{J+1}(r) = V_T(r)\, u_{J-1}(r)\,. \end{cases}$$

Let us remind first a few facts about scattering by tensor forces in general because, though these results are well-known by specialists, they are rather unfamiliar to the non-specialist.

(9) A. MARTIN and R. VINH MAU: *Nuovo Cimento*, **20**, 390 (1961).

When in eq. (32) the boundary condition $u_{J-1}(0) = u_{J+1}(r) = 0$ is disregarded, the system admits 4 independent solutions. Here we shall select the independent solutions by their asymptotic behaviour

$$(33) \qquad \begin{pmatrix} u_{J-1} \\ v_{J+1} \end{pmatrix} = \psi_1^+ \sim \begin{pmatrix} \exp[ikr] \\ \exp[ikr] \end{pmatrix}, \quad \psi_2^+ \sim \begin{pmatrix} \exp[ikr] \\ -\exp[ikr] \end{pmatrix}, \quad \psi_1^- \sim \begin{pmatrix} \exp[-ikr] \\ \exp[-ikr] \end{pmatrix},$$

$$\psi_2^- \sim \begin{pmatrix} \exp[-ikr] \\ -\exp[-ikr] \end{pmatrix}.$$

(Of course other linearly independent sets of solutions could be used). If we impose the boundary conditions at $r = 0$ we still have two linearly independent solutions.

The scattering matrix can be written as a two by two matrix:

$$(34) \qquad \begin{pmatrix} S_{J-1\,J-1} & S_{J-1\,J+1} \\ S_{J+1\,J-1} & S_{J+1\,J+1} \end{pmatrix},$$

$S_{J-1\,J-1}$ measures the amplitude of the outgoing $J-1$ wave when the initial $J-1$ ingoing wave is a pure $J-1$ wave; $S_{J-1\,J+1}$ measures the amplitude of the $J-1$ outgoing wave when one starts from a pure $J+1$ ingoing wave, and so on. So to construct the scattering matrix elements one has to fulfil at the same time certain boundary conditions at the origin and at infinity. For instance to determine $S_{J-1\,J-1}$ and $S_{J+1\,J-1}$ we should start from a combination

$$(35) \qquad a_1\psi_1^+ + a_2\psi_2^+ + c(\psi_1^- + \psi_2^-),$$

where no $L = J+1$ incoming wave is present. Then the $L = J-1$ asymptotic wave is

$$(36) \qquad (a_1 + a_2)\exp[ikr] + 2c\exp[-ikr],$$

hence, by comparison with $-S_{J-1\,J-1}\exp[ikr] + \exp[-ikr]$ we get

$$S_{J-1\,J-1} = -\frac{a_1 + a_2}{2c},$$

and also, clearly

$$S_{J+1\,J-1} = \frac{a_2 - a_1}{2c},$$

once $\psi_1^+\psi_1^+\psi_2^-\psi_2^-$ are known, a_1/c and a_2/c are determined by the condition

$$a_1\psi_1^+(0) + a_2\psi_2^+(0) + c[\psi_1^-(0) + \psi_2^-(0)] = 0\,, \tag{37}$$

which furnishes two equations (*).

The S matrix constructed in this way has to be unitary and symmetric: for real energies this means:

$$\begin{cases} S_{J-1\ J+1} = S_{J+1\ J-1}\,, \\ |S_{J+1\ J-1}|^2 + |S_{J-1\ J-1}|^2 = 1\,, \\ |S_{J+1\ J+1}|^2 + |S_{J-1\ J+1}|^2 = 1\,, \\ S_{J-1\ J-1}S^*_{J-1\ J+1} + S_{J+J\ J-1}S^*_{J+1\ J+1} = 0\,. \end{cases} \tag{38}$$

The latter equation is already satisfied in modulus if the three first equations are fulfilled. Therefore the 8 real variables appearing in the four matrix elements are connected by 5 independent linear equations and the scattering matrix can be expressed in terms of 3 real quantities.

This is probably the reason why many people prefer to express the information on the scattering in an other way where no superabundant quantities appear. In the representation above considered the S-matrix connects the ingoing states

$$\begin{pmatrix} \exp[-ikr] \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ \exp[-ikr] \end{pmatrix},$$

with the outgoing states

$$\begin{pmatrix} \exp[ikr] \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ \exp[ikr] \end{pmatrix}$$

One could try to choose another orthogonal set of ingoing and outgoing states such that the S matrix becomes diagonal, *i.e.* make an orthogonal transformation on S. We take the ingoing states

$$\begin{cases} U_\alpha^- \sim \dfrac{1}{\sqrt{1+\eta^2}} \begin{pmatrix} \exp[-ikr] \\ \eta \exp[-ikr] \end{pmatrix}, \qquad U_\beta^- \sim \dfrac{1}{\sqrt{1+\eta^2}} \begin{pmatrix} -\eta \exp[-ikr] \\ \exp[-ikr] \end{pmatrix}, \\ \text{and outgoing states} \\ U_\alpha^+ \sim \dfrac{1}{\sqrt{1+\eta^2}} \begin{pmatrix} \exp[ikr] \\ \eta \exp[ikr] \end{pmatrix}, \qquad U_\beta^+ \sim \dfrac{1}{\sqrt{1+\eta^2}} \begin{pmatrix} -\eta \exp[ikr] \\ \exp[ikr] \end{pmatrix}. \end{cases} \tag{39}$$

(*) The effect of the centrifugal term will be considered more seriously later.

then the unitarity condition remains valid after the transformation and if we choose η such that $S_{\alpha\beta} = S_{\beta\alpha} = 0$ then $|S_{\alpha\alpha}| = |S_{\beta\beta}| = 1$. η is determined by the equation

$$S_{J-1\ J+1}(1-\eta^2) + \eta(S_{J+1\ J+1} - S_{J-1\ J-1}) = 0\,, \tag{40}$$

which expresses the fact that η can take two values $\eta_\alpha\ \eta_\beta$ such that $\eta_\alpha \eta_\beta = -1$, which just corresponds to the two ingoing solutions we have considered. One can check, using the conditions from unitarity and symmetry that η is real.

Obviously once one has

$$S_{\alpha\alpha} = \exp[2i\delta_\alpha] \qquad \text{and} \qquad S_{\beta\beta} = \exp[2i\delta_\beta]\ (^*); \tag{41}$$

with η_α one can reconstruct the S matrix in its original form if one wishes.

For complex energies (if the S matrix makes sense), the *symmetry property* of the S matrix still holds, but the unitarity condition becomes $S^*(k^*)S(k) = 1$. It is still possible to diagonalize the S matrix; the equation for η still holds, with $\eta_\alpha \eta_\beta = -1$ but η_α becomes complex, and $|S_{\alpha\alpha}| \neq 1$, $|S_{\beta\beta}| \neq 1$.

To be complete we should add that in order to compute a cross-section, one has to adjust the amount of ingoing wave to what is present in an incident plane wave, and using the S matrix obtain the outgoing scattered wave.

Let us now return to our specific problem where $\psi_1^+ \psi_2^+ \psi_1^- \psi_2^-$ are obtained from potentials of the type

$$\begin{cases} V_{J-1} = \int_{\mu}^{\infty} C_{J-1}(\alpha) \exp[-\alpha r]\,\mathrm{d}\alpha\,, \\ V_{J+1} = \int_{\mu}^{\infty} C_{J+1}(\alpha) \exp[-\alpha r]\,\mathrm{d}\alpha\,, \\ V_T = \int_{\mu}^{\infty} C_T(\alpha) \exp[-\alpha r]\,\mathrm{d}\alpha\,. \end{cases} \tag{42}$$

Then we try to represent the wave functions as

$$\psi_1^- = \exp[-ikr] \begin{pmatrix} f_1(k,r) \\ \\ g_1(k,r) \end{pmatrix}, \qquad \psi_2^- = \exp[-ikr] \begin{pmatrix} f_2(k,r) \\ \\ g_2(k,r) \end{pmatrix}, \tag{43}$$

ψ_1^+ and ψ_2^+ are obtained by changing k to $-k$.

(*) δ_α, δ_β are called « eigenphases »; η_α, η_β are the « admixtures ».

We assume (as will be a posteriori checked to be right)

$$(44)\qquad \begin{cases} f_1(k,r) = 1 + \int_0^\infty \varrho_{1k}(\alpha)\exp[-\alpha r]\,\mathrm{d}\alpha\,, \\ f_2(k,r) = 1 + \int_0^\infty \varrho_{2k}(\alpha)\exp[-\alpha r]\,\mathrm{d}\alpha\,, \\ g_1(k,r) = 1 + \int_0^\infty \sigma_{1k}(\alpha)\exp[-\alpha r]\,\mathrm{d}\alpha\,, \\ g_2(k,r) = -1 + \int_0^\infty \sigma_{2k}(\alpha)\exp[-\alpha r]\,\mathrm{d}\alpha\,, \end{cases}$$

then we insert in the coupled Schrödinger equations

$$(45)\qquad \begin{cases} f'' - 2ikf' - \dfrac{(J-1)J}{r^2} f - V_{J-1} f = V_T\, g\,, \\ g'' - 2ikg' - \dfrac{(J+1)(J+2)}{r^2} g - V_{J+1} g = V_T\, f\,. \end{cases}$$

The equations can be treated exactly in the same way as in the $l \neq 0$ case. It turns out then that they can be transformed to be of the form

$$(46)\qquad \varrho_{1k}(\alpha) = \varrho_k^0(\alpha) + \psi(\alpha) + \int_0^\infty K_{J-1}(\alpha,\beta)\varphi(\beta)\,\mathrm{d}\beta\,,$$

and so on. Where $\varrho_k^0(\alpha) = (\mathrm{d}/\mathrm{d}\alpha) P_{J-1}(1 + (\alpha/ik))$. $\psi(\alpha)$ is known and vanishes for $\alpha < \mu$, and $\varphi(\beta)$, vanishing for $\beta < \mu$ contains $\varrho_{1k}(\gamma)$ and $\sigma_{1k}(\gamma)$ for arguments γ smaller than $\beta - \mu$. The iterative procedure works provided one solves at the same time the equation for $\varrho_{1k}(\alpha)$ and $\sigma_{1k}(\alpha)$. Eventually, one finds for $f_1(k,r)\,g_1 f_2 g_2$ the same analytic properties with respect to k as in the S wave case.

A slight difficulty occurs when one wants to construct from $f_1 g_1 f_2 g_2$ the S matrix. As it was sketched in the general considerations about tensor forces one has to take suitable combinations of $\psi_1^+ \psi_2^+ \psi_1^- \psi_2^-$ and to impose regularity conditions at the origin on the combination. Unfortunately, due to the centrifugal terms, $f_{1,2}$ and $g_{1,2}$ are singular at the origin. One would naively think that

$$\lim_{r\to 0} r^{J-1} f_1(k,r) \qquad \text{and} \qquad \lim_{r\to 0} r^{J-1} f_2(k,r)\,,$$

are finite, and in the same way

$$\lim_{r\to 0} r^{J+1} g_1(k, r) \qquad \text{and} \qquad \lim_{r\to 0} r^{J+1} g_2(k, r) .$$

If this were true it would not be difficult to apply the regularity condition at the origin. This is just what is done in the $l \neq 0$ uncoupled case. However, it was shown by H. CORNILLE ([10]) that this behaviour at the origin is only correct in one case, which is when the coupling potential $V_J(r)$ vanishes at the origin. This means that when $V_J(0) = 0$ there is no difficulty to construct $S_{J-1\,J-1}$, $S_{J-1\,J+1}$ and $S_{J+1\,J+1}$. If $V_J(0) \neq 0$ a serious trouble arises because one can find a set of four solutions, equivalent to the set $\psi_1^+\psi_2^+\psi_1^-\psi_2^-$, two of which are regular solutions, the other two being irregular solutions with *different* degrees of singularity at the origin. Any kind of limiting process to deal with finite quantities will project out the most singular part of $f_1 f_2 g_1 g_2$ and therefore the less singular of the irregular solutions will not appear. So the number of equations to determine the matrix elements will not be sufficient. A more elaborate procedure must be used: one must first take into account the worse singularity and then subtract out from $f_1 f_2 g_1 g_2$ the contribution of the most singular solution. This will give just the complementary equation needed. Let us notice that certain field theoretical potentials such as the Gartenhaus potential satisfy the condition $V_J(0) = 0$ so that the method can be applied without complication.

In any case the matrix elements $S_{J-1\,J-1}$, $S_{J+1\,J-1}$, $S_{J+1\,J+1}$ have the same analytic properties as the $S(k)$ function in the uncoupled case. They have the same cuts and further one can show that the poles in the upper half k plane lie on the imaginary axis and correspond to bound states.

The analytic properties of the diagonalized S matrix and of the admixture parameter are not simple. This comes from the fact that the diagonal matrix element, as well as the admixture are functions with two determinations. However, one can construct directly from $\psi_1^+\psi_2^+\psi_1^-\psi_2^-$ the eigensolutions: one starts from solutions

$$(\psi_1^- + A\psi_2^-) - S(\psi_1^+ + A\psi_2^+) ,$$

these are automatically eigensolutions. The two regularity conditions determine A (related to the admixture) and S. This may be useful in calculations for real energies.

([10]) H. CORNILLE: private communication. To be published.

3. – Application of the partial wave properties for generalized Yukawa potentials.

3·1. *Numerical calculations.* – The method we have developed in order to obtain the analytic properties of partial waves may be also considered as a practical method of calculation. Let us consider the S wave case, an approximate $S(k)$ will be obtained by taking

$$S_{\text{Approx}}(k) = \frac{1 + \int_{\mu}^{N\mu} \varrho_k(\alpha)\,\mathrm{d}\alpha}{1 + \int_{\mu}^{N\mu} \varrho_{-k}(\alpha)\,\mathrm{d}\alpha}, \tag{47}$$

and for real energies

$$\delta_{\text{Approx}}(k) = \operatorname{Arg}\left[1 + \int_{\mu}^{N\mu} \varrho_k(\alpha)\,\mathrm{d}\alpha\right]. \tag{48}$$

The question is to know what is the error committed.

When $C(\alpha) < M^{1-\eta}$, for real k for instance,

$$1 + \int_{\mu}^{\infty} |\varrho_k(\alpha)|\,\mathrm{d}\alpha < \exp\left[\frac{M}{\eta}\frac{1}{\mu^{\eta}}\right] \qquad (\text{case } \eta < 1),$$

and

$$\begin{cases} \left|\int_{N\mu}^{\infty} \varrho_k(\alpha)\,\mathrm{d}\alpha\right| < \int_{N\mu}^{\infty} M\,\frac{\mathrm{d}\alpha}{\alpha^{1+\eta}} \exp\left[\frac{M}{\eta}\frac{1}{\mu^{\eta}}\right] \\ \left|\int_{N\mu}^{\infty} \varrho_k(\alpha)\,\mathrm{d}\alpha\right| < \frac{M}{\eta}\frac{1}{(N\mu)^{\eta}} \exp\left[\frac{M}{\eta}\frac{1}{\mu^{\eta}}\right] = r, \end{cases} \tag{49}$$

r goes to zero as N goes to infinity.

Therefore

$$\delta(k) = \operatorname{Arg}\left[1 + \int_{\mu}^{N\mu} \varrho_k(\alpha)\,\mathrm{d}\alpha\right] \pm \sin^{-1}\frac{r}{\left|1 + \int_{\mu}^{N\mu} \varrho_k(\alpha)\,\mathrm{d}\alpha\right|}. \tag{50}$$

So an explicit bound can be put on the error committed.

As an application let us mention this small result if we take the Yukawa

potential: the condition to get a bound state at zero energy is

$$1 + \int_{\mu}^{\infty} \varrho_{k=0}(\alpha)\,\mathrm{d}\alpha = 0\,,$$

replacing this by

$$1 + \int_{\mu}^{3\mu} \varrho(\alpha)\,\mathrm{d}\alpha = 1 - \frac{V_0}{2\mu} - \frac{V_0}{6\mu} + \left(\frac{V_0}{\mu}\right)^2 \left[\frac{1}{3} - \log\frac{4}{3}\right] = 0\,,$$

we get a well depth parameter

$$\frac{\mu}{V_0} = 0.595\,,$$

to be compared with the exact result 0.5953.

For $l \neq 0$ or for tensor forces the practical use of the method is not so obvious. However, the method we use seems to be very suitable for a computer n view of its iterative character.

3·2. *The deuteron wave function and the anomalous threshold problem* [8,11-14]. – We have already said that our method furnishes an integral representation of the ingoing and outgoing wave functions. In the special case of a bound state the wave function $f(-i\varkappa, r)\exp[-\varkappa r]$ is the regular wave function because $f(-i\varkappa, 0) = 0$. Hence the deuteron wave function may be written (not the reduced one):

$$\psi_{\mathrm{d}}(r) = N_{\mathrm{d}} \frac{\exp[-\varkappa r]}{r}\left[1 + \int_{\mu}^{\infty} \varrho_d(\alpha)\exp[-\alpha r]\,\mathrm{d}\alpha\right]. \tag{51}$$

This is the most general form of the deuteron wave function for Yukawa-like potentials. In a sense it is a generalization of the Hulthén wave function

$$\psi_{\text{Hulthén}} = N \frac{\exp[-\varkappa r] - \exp[-\beta r]}{r}\,.$$

The analytic properties of any non-relativistic matrix element involving

(11) L. Bertocchi, C. Ceolin and M. Tonin: *Nuovo Cimento*, **18**, 770 (1960).
(12) R. Blanckenbeckler and Y. Nambu: *Nuovo Cimento*, **18**, 525 (1960)
(13) A. Martin: *Nuovo Cimento*, **19**, 344 (1961).
(14) A. Martin and R. Vinh Mau: *Nuovo Cimento*, **20**, 246 (1961).

a deuteron in its initial or final state can be investigated by inserting in it the deuteron wave function.

This has been applied to two cases:

i) The non-relativistic deuteron electromagnetic form factor

$$F(\boldsymbol{q}^2) = \int \psi_d(\boldsymbol{r}) \exp\left[\frac{i\boldsymbol{q}\boldsymbol{r}}{2}\right] \psi_d(\boldsymbol{r})\, d^3r \,. \tag{52}$$

Typical contributions to this will be obtained by replacing $|\psi_d(r)|^2$ by

$$\left(\frac{\exp[-\varkappa r]}{r}\right)^2, \qquad \frac{\exp[-2\varkappa r]\exp[-\alpha r]}{r^2} \qquad (\alpha > \mu)\,,$$

$$\exp[-\alpha r]\exp[-\beta r]/r^2 \qquad (\alpha, \beta > \varkappa + \mu)\,.$$

Let us call

$$\varphi(\alpha, \beta, \boldsymbol{q}^2) = -\int \exp\left[i\frac{\boldsymbol{q}\boldsymbol{r}}{2}\right] \frac{\exp[-\alpha r]\exp[-\beta r]}{r^2}\, d^3r \,,$$

$$\frac{d\varphi}{d\alpha} = -\int \exp\left[i\frac{\boldsymbol{q}\boldsymbol{r}}{2}\right] \exp[-(\alpha+\beta)r]\, d^3r \,,$$

$$= -\frac{1}{(\alpha+\beta)^2 + (\boldsymbol{q}/2)^2} \times \text{const} \,.$$

Hence

$$\varphi(\alpha, \beta, \boldsymbol{q}^2) = \text{const} \int_{\alpha}^{\infty} \frac{d\gamma}{(\gamma+\beta)^2 + (\boldsymbol{q}/2)^2} \,,$$

therefore in the (q^2) complex plane $\varphi(\alpha, \beta, (q^2))$ has a branch point at $q^2 = -4(\alpha+\beta)^2$.

From this we conclude that the analytic properties of $F(q^2)$ are as follows: it has a branch point at $q^2 = -16\varkappa^2 = -4MB$ (where B is the deuteron binding energy) and then a cut from $q^2 = -4(2\varkappa+\mu)^2$ to $q^2 = -\infty$ (see Fig. 3).

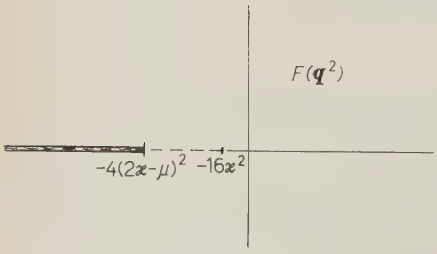


Fig. 3.

It can be noticed that the discontinuity across the cut starting from $q^2 = -16\varkappa^2$ (which is placed for convenience in the same direction as the next cut) can be calculated exactly in

the region

$$-16\varkappa^2 \to -4(2\varkappa+\mu)^2\,,$$

from the knowledge of $\varkappa$ and N alone, *i.e.* of the asymptotic form of the deuteron wave function.

If the $\mathcal{N}-\mathrm{p}$ potential is partly known or, more precisely if $C(\alpha)$ is known in the range

$$\mu < \alpha < N\mu\,,$$

(see next section) one can use our equations to calculate $\varrho_{-i\varkappa}(\alpha)$ and then the discontinuity of $F(q^2)$ across the cut till $q^2=-4(2\varkappa+(N+1)\mu)^2$.

It is interesting to compare the result of this analysis with the field-theoretical treatment. In this treatment one considers the deuteron as an elementary particle coupled to two nucleons. In perturbation theory the simplest process of interaction of a photon with the deuteron is given by Fig. 4.

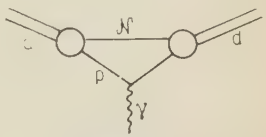


Fig. 4.

Let me remind you that the standard procedure to look for singularities with respect to q^2, the 4-momentum of the photon is to look at the lowest intermediate state in $\gamma \to \mathrm{d}+\bar{\mathrm{d}}$. The lowest intermediate state lies at $q^2=-(4\mu)^2$ one would believe that between $q^2=0$ and $q^2=-(4\mu^2)$ there is no singularity. However, if one calculates the singularities by explicit calculation of the Feynman denominator, as was done by KARPLUS, SOMMERFIELD and WICHMANN [15], it turns out that the first branch point is much closer to the physical region. It is called an *anomalous* threshold. With very good accuracy it lies at $q^2=-16\varkappa^2$, which is just (setting $q_0^2=0$, which is the case if we consider the form factor as a part of e-d scattering amplitude in c.m. system) the position at which we found the first branch point in the non-relativistic treatment. Therefore the appearance of anomalous thresholds is directly related to the weak binding or the large extension of the deuteron. If the deuteron binding energy were much bigger, the relativistic treatment predicts that the anomalous thresholds should disappear. The non-relativistic treatment is unable to give this result, and this for the simple reason that in a tightly bounded deuteron the non-relativistic quantum mechanics would no longer be valid.

ii) A similar but much more complicated case is the deuteron photodisintegration matrix element. We can write this matrix element, neglecting

[15] R. KARPLUS, C. SOMMERFIELD and M. WICHMANN: *Phys. Rev.*, **111**, 1187(1958); *Phys. Rev.*, **114**, 376 (1959).

spin complications as

$$\int \psi^{\boldsymbol{p}_f *}_{NP}(\boldsymbol{x}) \exp\left[i\frac{\boldsymbol{q}\boldsymbol{x}}{2}\right] \psi_{\mathrm{d}}(\boldsymbol{x})\, \mathrm{d}^3x\,, \tag{53}$$

where $\psi^{\boldsymbol{p}_f}_{NP}(\boldsymbol{x})$ is the Np wave function (incoming wave+plane wave) with c.m. momentum $\boldsymbol{p}_f$, and $\boldsymbol{q}$ is the photon momentum. One has to study the analytic properties of

$$\int \psi^{\boldsymbol{p}_f *}_{NP}(\boldsymbol{x}) \frac{\exp[-\alpha x]}{x} \exp\left[i\frac{\boldsymbol{q}\boldsymbol{x}}{2}\right] \mathrm{d}^3x\,,$$

where $\alpha = \varkappa_{\mathrm{d}}$ or $\alpha > \varkappa_{\mathrm{d}} + \mu$. This object is very similar to a scattering matrix element

$$\int \psi^{\boldsymbol{p}_f}_{NP}(\boldsymbol{x}) \exp[i\boldsymbol{p}_i\boldsymbol{x}] \left| \int_{\mu}^{\infty} C(\alpha) \frac{\exp[-\alpha x]}{x}\, \mathrm{d}\alpha \right| \mathrm{d}^3x\,.$$

As far as the analytic properties with respect to the angle $\boldsymbol{p}_f\boldsymbol{q}$ for fixed $\boldsymbol{p}_i$ and $\boldsymbol{q}$ are concerned, both matrix elements can be treated in the same way. One can just copy the results for the analytic properties of an off-energy shell ($|\boldsymbol{p}_i| \neq \boldsymbol{p}_f|$) scattering matrix element to get the analytic properties with respect to the angle.

We shall speak later about the fixed energy analytic properties. So we shall just give here the result for the deuteron photodisintegration matrix element:

$$\text{it has a pole at} \qquad (\boldsymbol{p}_f - \boldsymbol{q}/2)^2 = -\varkappa^2\,,$$

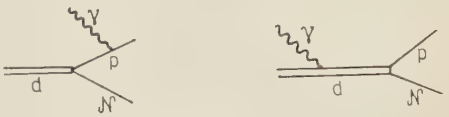
$$\text{and a cut starting at} \quad (\boldsymbol{p}_f - \boldsymbol{q}/2)^2 = -(\varkappa+\mu)^2\,.$$

The fixed transfer dispersion relations for this matrix element are not very easily established because of the complications due to kinematics. This, however, can be done [14] and by combination with the fixed energy dispersion relations for nearly real energies one gets a Mandelstam representation for this matrix element.

A much simpler thing to do is to look at the analytic properties of the partial waves for this matrix element [8]. Then one can also replace the final wave function by its integral representation derived in Section 2. Needless to say that the analytic properties obtained in this way coincide with those obtained by projecting the Mandelstam representation.

The field theoretical analogue of this matrix element is described by the

following graphs Fig. 5. The second order graphs correspond to the poles appearing in the energy or in the momentum transfer variables. The fourth order graph gives rise to anomalous thresholds in agreement with the non-relativistic treatment (cut at $(\boldsymbol{p}_f - \boldsymbol{q}/2)^2 = -(\varkappa + \mu)^2$).



It is interesting to notice that in spite of the presence of anomalous thresholds the Mandelstam representation holds, at least in the non-relativistic approach.

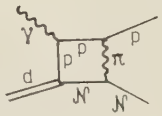


Fig. 5.

Other applications can be made using this technique, for instance to study the electromagnetic form factor of the α-particle ([16]), which can be considered either as a d-d system or a N-^{3}He system. Then one gets cuts due to the extension of the α-particle but one can show that the closest cut starts relatively far from the physical region ($q^2 = 11\mu^2$) and that it is possible to observe the effect of light intermediate particles, for instance a $T = 0$, $J = 1$ particle of mass $2.2\mu_\pi$, which gives rise to closer singularities (Fig. 6).

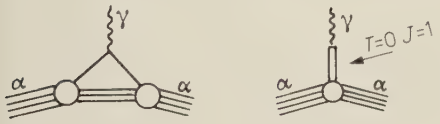


Fig. 6.

3·3. *The connection between the unphysical cut of a partial wave amplitude, the S matrix, and the potential* ([17]). – In recent years many theoreticians have attempted to relate the nucleon-nucleon scattering matrix to pion-pion and pion-nucleon scattering, using Mandelstam representation and unitarity. In many approaches, the primary object which is calculated or estimated is the discontinuity of the S matrix of a given wave on the unphysical cut in the energy complex plane. Once this discontinuity is known one has to construct a scattering amplitude such that it has the correct discontinuity on the left and fulfils the unitarity condition on the physical cut ([18]).

In the simple case of potential scattering (for Yukawa-like potentials) one would like to be able to construct the S scattering amplitude (or any other partial wave amplitude) from the information on the discontinuity of the left-hand cut in the k^2 (*i.e.* kinetic energy) plane (Fig. 7). This might prove

([16]) L. Bertocchi, S. Fubini and M. Tonin: private communication.

([17]) A. Martin: *Nuovo Cimento*, **19**, 1257 (1961); V. De Alfaro and T. Regge: *Nuovo Cimento*, **20**, 956 (1961).

([18]) H. P. Noyes and D. Y. Wong: *Phys. Rev. Lett.*, **3**, 191 (1959).

useful in the more general case, because, if we succeed in doing so, we may forget the intermediate assumption of the existence of a potential, and obtain an acceptable solution, as long as relativistic kinematical effects are unimportant.

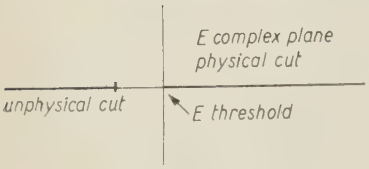


Fig. 7.

In our study of analytic properties of partial waves we have used k rather than k^2 as a variable: the upper half k plane corresponds to the first k^2 Riemann sheet (with respect to the physical cut). The cut $k = i\mu/2 \to i\infty$ is precisely the unphysical cut. The lower cut of the k plane is hidden in the second Riemann sheet of the energy complex plane.

So, what is supposed to be known is the discontinuity $S(ix+\varepsilon) - S(ix-\varepsilon)$ across the upper cut in the k plane:

$$S(ix+\varepsilon) - S(ix-\varepsilon) = D(x)\cdot 2\pi i\,, \tag{54}$$

where D is a real function.

We shall write

$$S(k) \quad \frac{f(k,0)}{f(-k,0)}\,,$$

as was previously done and remember that for potentials such that $C(\alpha) < \alpha^{1-\eta}$

$$\lim_{|k|\to\infty} \frac{|f(k,0)-1|}{|k|^{\eta}} < \text{const}\,.$$

This enables us to write an unsubtracted integral representation for f:

$$f(k,0) = 1 - i\int_{\mu/2}^{\infty} \frac{R(x)\,\mathrm{d}(x)}{k - ix}\,, \tag{55}$$

$R(x)$ is a real function because $f^*(-k^*, 0) = f(k, 0)$. We get

$$S(k) = \frac{1 - i\int_{\mu/2}^{\infty} R(x)\,\mathrm{d}x/(k-ix)}{1 + i\int_{\mu/2}^{\infty} R(x)\,\mathrm{d}x/(k+ix)} \tag{56}$$

it is a trivial matter to evaluate the discontinuity of S across the upper cut.

$$D(x) = -\frac{R(x)}{1 + \int_{\mu/2}^{\infty} R(x')\,\mathrm{d}x'/(x' + x)} \quad (*)\,, \tag{57}$$

or

$$R(x) = -D(x)\left[1 + \int_{\mu/2}^{\infty} R(x')\,\mathrm{d}x'/(x' + x)\right]. \tag{58}$$

This equation can be considered as a Fredholm (or almost Fredholm) type integral equation determining $R(x)$, and hence $S(k)$ from the information on the discontinuity. This equation was obtained by NOYES and WONG [18] for the nucleon-nucleon case, at least for the non-relativistic kinematics, in a rather involved way, as the result of the combination of two coupled integral equations, one expressing the fact that the left-hand cut has the correct discontinuity, the other expressing that the physical cut discontinuity (in the energy variable) satisfies unitarity.

Our method can be actually generalized to the general case where no potential is assumed. This has been done by Omnès [19]. One still writes

$$S(k) = \frac{\hat{f}(k)}{\hat{f}(-k)}\,,$$

$\hat{f}(k)$, as in the potential scattering case has no singularity in the lower half k plane, and goes to unity as $|k| \to \infty$. A natural definition for $\hat{f}(k)$ will be

$$\hat{f}(k) = \exp\left[-\frac{1}{\pi}\right]\int_{0}^{\infty} \frac{\delta(\nu')\,\mathrm{d}\nu'}{\nu' - \nu}\,,$$

(for k in the *lower half plane*), where ν is the kinetic energy. When the Levinson theorem is fulfilled and when no bound state is present (this imposes $\delta(0) - \delta(\infty) = 0$) $\hat{f}(k)$ coincides with our $f(k, 0)$. $\hat{f}(k)$ can be prolongated in the upper half plane by

$$\hat{f}(k) = S(\nu)\hat{f}(-k)\,.$$

(*) In ref. [17] the sign in *incorrect*.

[19] R. OMNÈS: *Nuovo Cimento* **21**, 524 (1961).

Then one shows that $\hat{f}(k)$ satisfies the same kind of integral representation as the one we proposed. The equation one ends up with is similar to ours but a little bit more complicated because:

i) of relativistic kinematics;

ii) the bound states no longer appear as zeros of $\hat{f}(-k)$ for k in the upper half plane but as poles of $\hat{f}(k)$;

iii) if $\delta(0)-\delta(\infty)\neq 0$, but rather $n\pi$, $\hat{f}(k)$ has a multiple pole at $k=0$.

Let us come back to our original equation. Its solutions have been studied by Regge and De Alfaro [17] who have shown that if

$$\frac{1}{2}\int_{\mu/2}^{\infty}\frac{|D(x)|}{x}\,\mathrm{d}x = A < \ln 2\,,$$

the solution is unique; also, if $D(x)$ is positive and A finite the solution exists. Further they show that the number of resonances+antibound states is less than two plus the number of zeros of $D(x)$, if the above condition is fulfilled.

The next question we shall consider now is the relation between the potential and the discontinuity $D(x)$. One might question about the usefulness of constructing the potential since we have seen that from $D(x)$ one can directly construct the S matrix. However, three points are worth noticing:

i) when the discontinuity is not completely known S cannot be constructed, but it will turn out that one can construct explicitly the external part of the potential;

ii) information on the external part of the potential permits to determine the external (however, inside the interaction region) part of the deuteron wave function which as it appeared in *b*) is so important for the deuteron form factor;

iii) nuclear physicists have to deal with potentials in the many-body problem.

We shall first calculate the discontinuity from the potential. In order to do this we have to look more carefully at what happens for $k=ix\pm\varepsilon$ (x real $>\mu/2$). $f(k,r)$ has no longer a unique meaning but we have some hope to define

$$f^{+}(ix,r)=\lim_{\varepsilon\to 0} f(ix+\varepsilon,r)\,, \tag{59}$$

and, similarly $f^{-}(ix,r)$. Correspondingly we define $\varrho_{ix}^{+}(\alpha)$ and $\varrho_{ix}^{-}(\alpha)$. Now we

notice that from the equation

$$\alpha(\alpha - 2x \pm i\varepsilon)\,\varrho_{ix}^{\pm}(\alpha) = C(\alpha) + \int_{\mu}^{\alpha-\mu} C(\alpha-\beta)\,\varrho_{ix}^{\pm}(\beta)\,\mathrm{d}\beta\,, \tag{60}$$

i) $\varrho_{ix}^{+}(\alpha) - \varrho_{ix}^{-}(\alpha) = 0$ for $\alpha < 2x$;

ii) near $\alpha = 2x$;

$$\varrho_{ix}^{+}(\alpha) - \varrho_{ix}^{-}(\alpha) = -2\pi i\,\delta(\alpha - 2x)\,\frac{C(\alpha) + \int_{\mu}^{\alpha-\mu} C(\alpha-\beta)\,\varrho(\beta)\,\mathrm{d}\beta}{\alpha}\,. \tag{61}$$

There is no ambiguity in the right-hand side because of i).

iii) in the range $2x + \varepsilon < \alpha < 2x + \mu - \varepsilon$, $\varrho_{ix}^{+}(\alpha) - \varrho_{ix}^{-}(\alpha) = 0$.

Now there are several possible ways of calculating the discontinuity of $S(k)$

$$S^{+}(ix) - S^{-}(ix) = 2\pi i D(x) = \frac{\int[\varrho_{ix}^{+}(\alpha) - \varrho_{ix}^{-}(\alpha)]\,\mathrm{d}\alpha}{1 + \int_{\mu}^{\infty}\varrho_{-ix}(\alpha)\,\mathrm{d}\alpha}\,. \tag{62}$$

The most straightforward may consist in noticing that $f^{+}(ix, r)\exp[xr]$, $f^{-}(ix, r)\exp[xr]$, $f(-ix, r)\exp[-xr]$ are not independent solutions of the Schrödinger equation. The only possible relation between these quantities, if we take into account the fact that

$$f^{+}(ix, \infty) = f^{-}(ix, \infty) = f(-ix, \infty) = 1\,,$$

is

$$f^{+}(ix, r) - f^{-}(ix, r) = 2i\pi A(x)\exp[-2xr](-ix, r)\,.$$

If we look at this relation for $r \to 0$ we see that $A(x)$ is identical with $D(x)$. So we can evaluate $D(x)$ by taking the most suitable value of r; in particular

$$2\pi i D(x) = \lim_{r\to\infty}\exp[2xr][f^{+}(ix, r) - f^{-}(ix, r)] = \tag{63}$$

$$= \lim_{r\to\infty}\exp[-[2xr]\int_{2x-\varepsilon}^{2x+\varepsilon}[\varrho_{ix}^{+}(\alpha) - \varrho_{ix}^{-}(\alpha)]\exp[-\alpha r|\,\mathrm{d}\alpha +$$

$$+ \lim_{r\to\infty}\exp[-2xr]\int_{2x+\mu-\varepsilon}^{\infty}[\varrho_{ix}^{+}(\alpha) - \varrho_{ix}^{-}(\alpha)]\exp[-\alpha r]\,\mathrm{d}\alpha.$$

The last term does not contribute to $D(x)$ because it contains a factor $\exp[-(\mu-\varepsilon)r]$. The contribution of the first term is trivially evaluated. It gives just

$$D(x) = -\frac{C(2x) + \int_{\mu}^{2x-\mu} C(2x-\alpha)\,\varrho_{ix}(\alpha)\,\mathrm{d}\alpha}{2x} \,. \tag{64}$$

In this formula $\varrho_{ix}(\alpha)$ is a well defined quantity because the range of integration is such that the condition $\alpha < 2x$ is always fulfilled. At this point we have really proved that the lines

$$k = i\frac{\mu}{2} \to i\infty \qquad \text{and} \qquad k = -i\frac{\mu}{2} \to -i\infty \,,$$

are cuts of the function $\exp[2i\delta(x)]$. We have an explicit expression of the discontinuity. One can notice that if the potential is written

$$V(r) = \lambda \int_{\mu}^{\infty} C(\alpha)\exp[-\alpha r]\,\mathrm{d}\alpha \,,$$

the discontinuity in the range $x = n(\mu/2)$, $x = (n+1)(\mu/2)$ is given by an n-th order polynomial in λ. Therefore it is just $S(ix+\varepsilon) - S(ix-\varepsilon)$ calculated till the n^{th} Born approximation. However, the expression we have derived is exact, even when the Born series fails to converge.

It is interesting to notice that an attractive potential with $C(\alpha) < 0$ gives rise to a *purely positive* $D(x)$. Indeed looking at

$$\alpha(\alpha - 2x)\,\varrho_{ix}(\alpha) = C(\alpha) + \int_{\mu}^{\alpha-\mu} C(\alpha-\beta)\,\varrho_{ix}(\beta)\,\mathrm{d}\beta \,,$$

we easily see that $C < 0$ makes $\varrho_{ix}(\alpha) > 0$ for $\alpha < 2x$. Inserting this in eq. (64) we see that $D(x)$ has to be *positive*. Returning to eq. (58) we see that if there is no bound state with binding energy larger than $B = (\mu^2/4)/M$ (which is the case of the $\mathcal{N}$p system)

$$1 + \int_{\mu/2}^{\infty} \frac{R(x')\,\mathrm{d}x'}{x'+x} \,,$$

is positive for $x > \mu/2$, and hence $R(x)$, if $D(x)$ is positive, is purely negative.

Then

$$1 + \int_{\mu/2}^{\infty} \frac{R(x')\,\mathrm{d}x'}{x' + x} ,$$

can only vanish once in the range $x = 0$, $x = \mu/2$ and there is at most one bound state. Hence, if a system interacting by a purely attractive potential has two bound states one of these bound states must have a binding energy larger than $(\mu^2/4)/M$. Let us also notice that if $R(x)$ is strictly negative there is no zero of $f(k, 0)$ outside the imaginary axis because then $f(k, 0)$ is a Herglotz function (Im f has the sign of Re k). This means that there is no resonance since there is no pole close to the real axis in the lower complex k half plane There may be, however, a pole on the imaginary axis between $k = 0$ and $k = -i(\mu/2)$ which would correspond to a « virtual state » (as in p-p system), but then the true bound state cannot exist.

Let us now exploit formula (64) in order to construct the potential from the information on the discontinuity.

If we know $D(x)$ in the range $x = \mu/2 \to x = 2(\mu/2)$ we get

$$C(2x) = -2x\,D(x) .$$

Let us go now to the range $x = \mu$, $x = 3\mu/2$ we know from the preceding result that for $\alpha < 2\mu$

$$\varrho_{ix}(\alpha) = \frac{C(\alpha)}{\alpha(\alpha - 2x)} = \frac{D(\alpha/2)}{2x - \alpha} .$$

Hence

$$-2xD(x) = C(2x) + \int_{\mu}^{2x-\mu} \frac{C(2x - \beta)\,D(\beta/2)\,\mathrm{d}\beta}{2x - \beta} = C(2x) - \int_{\mu}^{2x-\mu} D(x - \beta/2)\,D(\beta/2)\,\mathrm{d}\beta ,$$

or

$$C(2x) = -2xD(x) + \int_{\mu}^{2x-\mu} D(x - \beta/2)\,D(\beta/2)\,\mathrm{d}\beta .$$

So, here again we are able to construct $C(\alpha)$ from D.

The procedure can go on: if we know $C(\alpha)$ in the range $\alpha = \mu$, $\alpha = n\mu$ and $D(x)$ in a larger range, say

$$x = \mu/2 \qquad \text{to} \qquad x = \frac{n+1}{2}\,\mu .$$

We know $\varrho_{ix}(\alpha)$ for $\alpha < n\mu$ and we can calculate the integral appearing in the r.h.s. of (64); the l.h.s. is known by hypothesis; hence $C(2x)$ can be determined for $n\mu < 2x < (n+1)\mu$.

In general, when the discontinuity is known for $2x < N\mu$ the contribution to the potential

$$\int_{\mu}^{N\mu} C(\alpha) \exp[-\alpha r] \mathrm{d}\alpha ,$$

is known.

We see that the further the discontinuity is known, the better the inner part of the potential is known. If the discontinuity is completely known and if the conditions imposed by Regge and De Alfaro are fulfilled, it seems that the complete potential

$$\int_{\mu}^{\infty} C(\alpha) \exp[-\alpha r] \mathrm{d}\alpha ,$$

is acceptable because we know anyway from previous work by GEL'FAND, LEVITAN and others that one can fit the phase shift by a potential.

The condition that the maximum range of the potential is $1/\mu$ is a very important one. It permits, as long as there are no bound states with $x > \mu/2$ to distinguish between the zeros of $f(-k, 0)$ and the singularities of $f(k, 0)$. This is the reason why in our treatment we do not get various phase equivalent potentials.

In case a hard core is present, the treatment as given here breaks down because the scattering amplitude has an essential singularity at infinity. However, if the range of the hard core is known the procedure can be adapted to this case.

One starts from

$$V(r) = +\infty \qquad \text{for} \qquad r_0 > r ,$$

$$V(r) = \int_{\mu}^{\infty} C(\alpha) \exp[-\alpha(r - r_0)] \mathrm{d}\alpha \qquad \text{for} \qquad r_0 < r ,$$

one defines

$$S'(k) = S(k) \exp[2ikr_0] ,$$

and

$$D'(x) = D(x) \exp[-2xr_0] ,$$

from $D'(x)$ one can construct $C(\alpha)$ in the ordinary way.

The connection between the discontinuity and the potential can be extended to an arbitrary partial wave. Formula (64) can be written as

$$D(x) = -\lim_{\alpha \to 2x} \varrho_{ix}(\alpha)(\alpha - 2x) \,. \tag{65}$$

Then the expression of $D(x)$ can be obtained in terms of $C(\alpha)$ and $\varrho(\alpha)$ which itself can be calculated from $C(\alpha)$. When one tries to go the reverse way, one finds that at each step of the calculation one must solve a Volterra equation to construct C from D. In this latter case it is clear that D must satisfy stringent conditions to ensure that the phase shift has a correct low energy behaviour. These conditions are not apparent in our treatment because they can always be satisfied by a convenient choice of $D(x)$ for large x.

The treatment can as well be extended to the construction of potentials in presence of tensor forces. Starting from the discontinuities $D_{00} D_{02} D_{22}$ of $S_{00}(k) S_{02}(k) S_{22}(k)$ one can construct the inverse Laplace transform of V_0, V_2, V_{tensor}.

4. – Analytic properties for fixed energy [4].

The complete scattering amplitude is a function of two variables $f(E, t)$, where $t = -2k^2(1 - \cos\theta)$, k is the c.m. momentum and θ the scattering angle, and $E = k^2$ is the energy.

The most straightforward approach to the analytic properties for fixed physical E with respect to t is to make use of the Born expansion. However, it will turn out eventually that the analytic properties of the exact amplitude can be obtained from those derived by using the Born expansion in a very easy way. The only question this approach cannot solve is the behaviour of $f(E, t)$ for $t \to \infty$. Among all the approaches to this problem only the one by Regge furnishes the solution. However, this does not mean that the present approach is useless, because it gives another insight of the structure of the discontinuities.

For convenience we assume for the potential

$$V(r) = \int_{\mu}^{\infty} \Gamma(\alpha) \frac{\exp[-\alpha r]}{r} \, \mathrm{d}\alpha \,,$$

i.e. we take it as a superposition of Yukawa potentials rather than exponential potentials.

The n-th Born term can be written, dropping numerical factors:

$$\langle \boldsymbol{k}' \,|\, V G_{0k} V G_{0k} \ldots V \,|\, \boldsymbol{k} \rangle , \qquad \text{with } n \text{ } V\text{'s}, \tag{67}$$

and

$$\begin{cases} G_{0k} = \dfrac{\exp\,[ik\,|r - r'|]}{|r - r'|}, & \text{in coordinate space,} \\ \quad = \dfrac{k'^2 - k^2 - i\varepsilon}{1}, & \text{in momentum space.} \end{cases} \tag{68}$$

Inserting the representation for $V(r)$ we get a typical contribution to the n-th Born term:

$$\int k_1^2 \,\mathrm{d}k_1 \,\mathrm{d}\Omega_1 \ldots \frac{1}{\alpha_1^2 + (\boldsymbol{k}' - \boldsymbol{k}_1)^2} \frac{1}{k_1^2 - k^2 - i\varepsilon} \frac{1}{\alpha_2^2 + (\boldsymbol{k}_1 - \boldsymbol{k}_2)^2} \cdots \frac{1}{\alpha_n^2 + (\boldsymbol{k}_{n-1} - \boldsymbol{k})^2}, \tag{69}$$

with $\alpha_1, \alpha_2, \ldots, \alpha_n > \mu$ this formula should be integrated again on $\alpha_1, \ldots, \alpha_n$, with the weight function $\Gamma(\alpha)$.

To study the analytic properties with respect to the angle $(\boldsymbol{k}', \boldsymbol{k})$ which is equivalent to the study of the analyticity in t we just need to make the angular integrals. Let us consider first the integration over k_1.

$$\int \mathrm{d}\Omega_1 \frac{1}{\alpha_1^2 + (\boldsymbol{k}' - \boldsymbol{k}_1)^2} \frac{1}{\alpha_2^2 + (\boldsymbol{k}_1 - \boldsymbol{k}_2)^2} = \int \frac{\mathrm{d}\Omega_1}{4k'k_2k_1^2} \frac{1}{U_1 - (\hat{k}', \hat{k}_1)} \frac{1}{U_2 - (\hat{k}_1, \hat{k}_2)},$$

with

$$U_1 = \frac{a_1^2 + k'^2 + k_1^2}{2k'k_1}, \qquad U_2 = \frac{\alpha_2^2 + k_1^2 + k_2^2}{2k_1k_2},$$

and the $\hat{}$ mean unit vectors.

This integral by means of Feynman parametrization can be transformed into

$$-\frac{2\pi}{4k'k_2k_1^2} \int_{-1}^{+1} \frac{\mathrm{d}y}{\left| U_1\left(\frac{1+y}{2}\right) + U_2\left(\frac{1-y}{2}\right)\right|^2 - \frac{1+y^2}{2} - \frac{1-y^2}{2}\,(\hat{k}', \hat{k}_2)},$$

or

$$-\frac{2\pi}{4k'k_2k_1^2} \int_{-1}^{+1} \frac{\mathrm{d}y}{\left(\frac{1-y^2}{2}\right)\left[\frac{[U_1\,((1+y)/2) + U_2\,((1-y)/2)]^2 - (1+y^2)/2}{(1-y^2)/2} - (\hat{k}', \hat{k}_2)\right]}.$$

Let us rewrite the bracket as $U_{12}-\hat{k}'\hat{k}_2$ and minimize it first with respect to $|\boldsymbol{k}_1|$ (since k_1 goes from 0 to ∞)

$$U_{12}=\frac{\left[\frac{1}{4k_1}\left[(1+y)\,\frac{\alpha_1^2+k'^2}{k'}+(1-y)\,\frac{\alpha_2^2+k_2^2}{k_2}\right]+\frac{k_1}{4}\left[\frac{(1+y)}{k'}+\frac{(1-y)}{k_2}\right]\right]^2-\frac{1+y^2}{2}}{(1-y^2)/2}$$

Now since $(A/X)+BX\geqslant 2\sqrt{AB}$

$$U_{12}>\frac{\frac{1}{2}\left[(1+y)\,\frac{\alpha_1^2+k'^2}{k'}+(1-y)\,\frac{\alpha_2^2+k_2^2}{k_2}\right]\left[\frac{1+y}{k'}+\frac{1-y}{k_2}\right]-(1+y^2)}{1-y^2}.$$

Letting $t=(1+y)/(1-y)$ this becomes

$$U_{12}>\frac{1}{2}\left\{\frac{\alpha_1^2+k'^2}{k'}\,t+\frac{\alpha_2^2+k_2^2}{k_2}\right\}\left\{\frac{1}{k'}+\frac{1}{tk}\right\}-\frac{1}{2}\left(t+\frac{1}{t}\right),$$

which, again can be minimized with respect to t. Eventually

$$U_{12}>\frac{(\alpha_1+\alpha_2)^2+k^2+k_2^2}{2kk_2}.$$

Therefore

$$U_{12}=\frac{(\lambda_{12}+\alpha_1+\alpha_2)^2+k^2+k_2^2}{2kk_2},$$

with $\lambda_{12}>0$; λ_{12} is a function of y and k_1. U_{12} has the same form as U_1 and U_2.

We can now perform the integration on $\mathrm{d}\Omega_2$. We have to integrate

$$\frac{1}{U_{12}-\hat{k}'\hat{k}_2}\,\frac{1}{U_3-\hat{k}_2\hat{k}_3};$$

obviously the same reasoning can be applied. Eventually the n-th Born term will be written as

$$\int\frac{\mathrm{d}(\text{parametric variables})\,\mathrm{d}(\text{lengths of } k\text{'s})\times\text{Green's functions}\times 1}{\text{products of }((1-y^2)/2)\times[U_{123\ldots n}-\hat{k}'\hat{k}]},$$

with

$$U_{12\ldots n}>\frac{(\alpha_1+\alpha_2\ldots+\alpha_n)^2+k'^2+k^2}{2kk'}>\frac{n^2\mu^2+k^2+k'^2}{2kk'}.$$

The remaining integrations give no trouble because they do not give any trouble in the physical range (*).

The structure of the denominator exhibits a cut in the $\cos\theta$ plane from

$$\cos\theta = \frac{n^2\mu^2 + k'^2 + k^2}{2kk'} \quad \text{to} \quad +\infty\,, \tag{70}$$

or in $t = -(\boldsymbol{k}' - \boldsymbol{k})^2$, from $t = n^2\mu^2$ to $+\infty$.

Notice that we do not assume $|\boldsymbol{k}| = |\boldsymbol{k}'|$. Notice also that the result holds if the k^2 in the free Green's function is different from k and k'. Let us now consider the full scattering amplitude

$$T = T_1 + T_2 + \ldots + T_n + T_{n+1} + R_n\,, \tag{71}$$

where

$$R_n = \int (T - T_1)(\boldsymbol{k}', \boldsymbol{k}'', k)\, G_{0k}\, T_n(\boldsymbol{k}', \boldsymbol{k}, k)\, \mathrm{d}^3k''\,, \tag{72}$$

in coordinate space

$$(T - T_1)(\boldsymbol{k}', \boldsymbol{k}'', k) = \int \exp[-\boldsymbol{k}'\boldsymbol{x}]\, V(x)\, G_k(xx')\, V(x')\, \exp[i\boldsymbol{k}''\boldsymbol{x}']\, \mathrm{d}^3x\, \mathrm{d}^3x'\,,$$

where G_k is the exact (not the free) Green's function. $G_k(x, x')$ is regular and bounded by $\mathrm{const}/(|x' - x|)$ in the upper complex half plane in k. This still holds for real k. If $|V(x)| < Ct \exp[-\mu x]$ which is the case here, the convergence of this integral is dependent on the factor

$$\exp[-i\boldsymbol{k}'\boldsymbol{x}' - \mu|\boldsymbol{x}'| + i\boldsymbol{k}''\boldsymbol{x} - \mu|\boldsymbol{x}|]\,.$$

Let us take

$$\boldsymbol{k}'' = \boldsymbol{U}\cos\lambda\theta + \boldsymbol{V}\sin\lambda\theta\,,$$

$$\boldsymbol{k}' = \boldsymbol{U}\cos(1-\lambda)\theta + \boldsymbol{V}\sin(\lambda - 1)\theta\,,$$

with $\boldsymbol{UV}$ orthogonal unit vectors. Then the real part of the argument of the exponential is at most

$$|x|\left[k''\sqrt{(\mathrm{Im}\cos\lambda\theta)^2 + (\mathrm{Im}\sin\lambda\theta)^2} - \mu\right] +$$

$$+ |x'|\left[k'\sqrt{[\mathrm{Im}\cos(1-\lambda)\theta]^2 + [\mathrm{Im}\sin(1-\lambda)\theta]^2} - \mu\right],$$

(*) The singularities at $y = \pm 1$ may be shown to be harmless.

$\cos \lambda\theta = \cos \lambda(\theta_1 + i\theta_2) = \cos \lambda\theta_1 \cosh \lambda\theta_1 - i \sin \lambda\theta_1 \sinh \lambda\theta_2$; to make the first bracket negative we get

$$k'' \sinh \lambda\theta_2 - \mu < 0 ,$$

and the second

$$k' \sinh (1 - \lambda)\theta_2 - \mu < 0 .$$

Therefore the integral exists in the *smaller* of the ellipses in the $\cos \theta$ plane with foci -1, $+1$ and semi major axis

$$\left| \begin{array}{l} \cosh \theta_2 = \cosh \dfrac{1}{\lambda} \sinh^{-1} \dfrac{\mu}{k''} , \\ \cosh \theta_2 = \cosh \dfrac{1}{1 - \lambda} \sinh^{-1} \dfrac{\mu}{k'} . \end{array} \right.$$

The maximum domain is obtained when these ellipses coincide, by convenient choice of λ.

Then

$$\theta_2 = \sinh^{-1} \frac{\mu}{k'} + \sinh^{-1} \frac{\mu}{k''} ,$$

and the semi major axis is

$$a = \sqrt{1 + \frac{\mu^2}{k'^2}} \sqrt{1 + \frac{\mu^2}{k''^2}} + \frac{\mu^2}{k'k''} ,$$

This means that $T(k'k'', k)$ can be expanded in Legendre polynomials

$$T(\boldsymbol{k}', \boldsymbol{k}'', k) = \sum A_l P_l(\hat{k}', \hat{k}'') ,$$

with

$$\limsup_{l \to \infty} (A_l)^{1/l} \leqslant \frac{1}{a + \sqrt{a^2 - 1}} .$$

Now $T_n(\boldsymbol{k}'', \boldsymbol{k}, k)$ can also be expanded in Legendre polynomials because from its analytic structure this expansion is certainly convergent in an ellipse with foci -1, $+1$ and semi major axis

$$b = \frac{n^2\mu^2 + k^2 + k''^2}{2kk''} .$$

Therefore

$$T_n(\boldsymbol{k}'', \boldsymbol{k}, k) = \sum B_l P_l(\hat{k}'', \hat{k}) ,$$

with

$$\limsup_{l\to\infty} (B_l)^{1/l} = \frac{1}{b+\sqrt{b^2-1}}\,.$$

One can insert these expressions in the expression of K_n. The angular integrations can be performed because

$$\int P_l(\hat{k}'\hat{k}'')\,P_l(\hat{k}''\hat{k})\,\mathrm{d}\Omega_{\hat{k}''} = d_{ll'}\,\frac{4\pi}{2l+1}\,P_l(\hat{k}'\hat{k}\quad.$$

Hence

$$R_n = \int \frac{k'^2\,\mathrm{d}k'}{k'^2-k^2}\,\sum \frac{4\pi A_l{}^{B_l}}{2l+1}\,P_l(\hat{k}'\hat{k})\,.$$

The expansion under the integral is convergent inside an ellipse with foci -1, $+1$ whose semi major axis c is just given by

$$\frac{1}{c+\sqrt{c^2-1}} = \limsup (A_lB_l)^{1/l} = \frac{1}{a+\sqrt{a^2-1}}\,\frac{1}{b+\sqrt{b^2-1}}\,,$$

or

$$c = ab + \sqrt{a^2-1}\,\sqrt{b^2-1}\,.$$

Now

$$c = \cosh\left[\sinh^{-1}\frac{\mu}{k''} + \sinh^{-1}\frac{\mu}{k'} + \cosh^{-1}\frac{n^2\mu^2+k^2+k''^2}{2kk''}\right].$$

We have to minimize it with respect to k''. Since cosh and sinh are monotonous we can minimize first

$$\sinh\left[\sinh^{-1}\frac{\mu}{k''} + \cosh^{-1}\frac{n^2\mu^2+k^2+k''^2}{2kk''}\right].$$

If

$$X = \frac{\mu}{k''}\,,\qquad Y = \frac{n^2\mu^2+k^2+k''^2}{2kk''}\,.$$

This quantity is $XY + \sqrt{X^2+1}\,\sqrt{Y^2-1}$ but this is just the minimum with respect to $\lambda>0$ of

$$\frac{1}{2}\left[\left(\lambda X + \frac{Y}{\lambda}\right) + \lambda^2 - \frac{1}{\lambda^2}\right].$$

We minimize first this with respect to k'': it gives

$$\frac{1}{2}\left|4\left(\lambda\mu+\frac{k^2+n^2\mu^2}{2\lambda k}\right)\left(\frac{1}{2\lambda k}\right)+\lambda^2-\frac{1}{\lambda^2}\right|-\frac{1}{2}\left[\frac{2\mu}{k}+\lambda^2+\frac{n^2\mu^2}{k^2\lambda^2}\right]\geqslant\frac{(n+1)\mu}{k}\,.$$

Hence

$$c_{\text{minimum}}=\sqrt{1+\frac{\mu^2}{k'^2}}\sqrt{1+\frac{(n+1)^2\mu^2}{k^2}}+\frac{(n+1)\mu^2}{kk'}\,. \tag{73}$$

The region of regularity of R_n is an ellipse with foci -1, $+1$ and semi-major axis c increasing with n. One can make this ellipse arbitrarily large. Inside this ellipse the only singularities are those given by the Born expansion, *i.e.* a series of cuts starting at

$$\cos\theta=\frac{k^2+k'^2+\mu^2}{2kk'}\ \ldots\ \cos\theta=\frac{k^2+k'^2+n^2\mu^2}{2kk'}\ \ldots$$

$$(\text{or } t=\mu^2\ \ldots\ t=n^2\mu^2\ \ldots)\,.$$

We have now proven that outside the cuts there is no singularity. The only question we cannot answer is the problem of the behaviour at infinity. As we increase the dimensions of the ellipse we must take into account more and more Born terms and it looks impossible to obtain in this way an idea on the asymptotic behaviour.

At a given point of the cut the discontinuity can be in principle calculated since it consists of a finite number of contributions. As was done in the case of the analytic properties of the partial wave, one can here also try to construct $\Gamma(\alpha)$, the weight function appearing in the potential from the discontinuity in the t plane ([20]). The reasoning is very similar to the one we have already done: in $t=\mu^2$ $t=4\mu^2$ the only contribution to the discontinuity comes from the first Born term

$$T_1=\text{const.}\int\frac{\Gamma(\alpha)\,\mathrm{d}\alpha}{t-\alpha^2}\,.$$

The discontinuity is easily seen to be proportional to $\Gamma(\sqrt{t})/\sqrt{t}$. Therefore $\Gamma(\alpha)$ can be constructed in the range $\alpha=\mu$, $\alpha=2\mu$ because T_2 and the remaining terms have singularities starting at least at $t=4\mu^2$. Let us now generalize this.

Let us write the potential as

$$V=V_{1n}+V_{n\,n+1}+V_{n+1\,\infty}\,,$$

([20]) A. Martin and Gy. Targonski: to be published in *Nuovo Cimento*.

where

$$V_{1n} = \int_{\mu}^{n\mu} \Gamma(\alpha) \frac{\exp[-\alpha r]}{r} \, d\alpha , \qquad V_{nn+1} = \int_{n\mu}^{(n+1)\mu} \Gamma(\alpha) \frac{\exp[-\alpha r]}{r} \, d\alpha ,$$

$$V_{n+1\infty} = \int_{(n+1)\mu}^{\infty} \Gamma(\alpha) \frac{\exp[-\alpha r]}{r} \, d\alpha .$$

Assume now that we have been able to construct V_{1n}. We want to get $V_{n\,n+1}$ from the discontinuity of T in the range $t = n^2\mu^2$ $t = (n+1)^2\mu^2$. In this range the discontinuity is given by

$$\Delta T_1 + \Delta T_2 + \ldots + \Delta T_n .$$

Now the terms in which $V_{n+1\infty}$ appear do not contribute to the discontinuity because they give rise to a cut starting at least at $t = (n+1)^2\mu^2$. The contributions $T_2 + \ldots + T_n$ in which $V_{n\,n+1}$ appears not to contribute to the discontinuity either, because they are combined at least with one potential of range $1/\mu$ and the singularities start at least at $t = (\mu + n\mu)^2 = (n+1)^2\mu^2$. Therefore the discontinuity ΔT is given by

$$\Delta T = \Delta T_1 + \Delta(T_2 + \ldots + T_n)(V_{1n}) ,$$

where $T_2 \ldots T_n$ are calculated from V_{1n} alone which is supposed to be known. This equation determines ΔT_1 in the range $t = n^2\mu^2$, $t = (n+1)^2\mu^2$ and, because of the special form of T_1 it determines $\Gamma(\alpha)$ in the range $\alpha = n\mu$, $\alpha = (n+1)\mu$. One can then iterate. Eventually one can construct $\Gamma(\alpha)$ from $\alpha = \mu$ to $\alpha = N\mu$ if ΔT is known in the range $t = \mu^2 \to t = N^2\mu^2$. Here again the farther we know the discontinuity the better we know the potential.

This procedure furnishes an unambiguous definition of an energy-dependent potential fitting at a given energy a scattering amplitude having the properties implied by Mandelstam representation. In the low energy region, for the nucleon-nucleon case, FUBINI, CHARAP and TAUSNER ([21]) have shown that at least the external part of the potential (one and two-pion exchange) is energy-independent.

A possible application of this is the study of the range of the imaginary potential in nucleon-antinucleon scattering ([22]). For this one makes the pre-

([21]) J. CHARAP and S. FUBINI: *Nuovo Cimento*, **14**, 540 (1949); **15**, 73 (1960); J. CHARAP and J. TAUSNER: *Nuovo Cimento*, **18**, 316 (1960).

([22]) A. MARTIN: CERN preprint. To be published in *Phys. Rev.*

liminary remark that a purely elastic scattering amplitude (no other channel open) will be automatically associated with a purely real potential. In this case the unitarity condition reads

$$\int \psi^*_{\boldsymbol{k}'}(x)\,\mathrm{Im}\,[V(x)]\psi_{\boldsymbol{k}}(x)\,\mathrm{d}^3x \equiv \varphi(t) = 0\,,$$

for $|\boldsymbol{k}| = |\boldsymbol{k}'|$. $\varphi(t)$ can be easily seen to have the same analytic properties as the scattering amplitude in t; since it has to vanish in the physical range it vanishes everywhere and one concludes from this that $\mathrm{Im}\ V(x) = 0$ (assuming for $\mathrm{Im}\ V(x)$ the same representation as for $V(x)$).

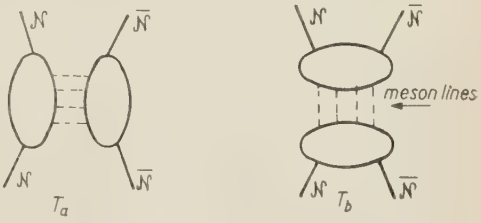


Fig. 8.

We can obviously apply our formalism to $\mathcal{N}\bar{\mathcal{N}} \to \mathcal{N}\bar{\mathcal{N}}$, but then the potential we get is clearly complex. Two kinds of graphs contribute to $\mathcal{N}\bar{\mathcal{N}} \to \mathcal{N}\bar{\mathcal{N}}$. T_a is purely elastic below the meson production threshold, *i.e.* if we use an energy such that the reaction $\mathcal{N}\bar{\mathcal{N}} \to \mathcal{N}\bar{\mathcal{N}} + \pi$ is impossible. T_a has the following analytic structure in the $\cos\theta$ plane (see Fig. 8 and 9).

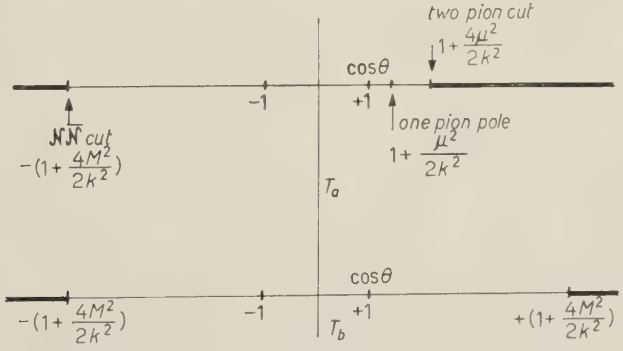


Fig. 9.

The singularities of T_b are symmetrical.

With such analytic properties we have to use a potential

$$\int \Gamma_{\mathrm{d}}(\alpha)\,\frac{\exp[-\alpha r]}{r}\,\mathrm{d}\alpha + P\int \Gamma_{\mathrm{e}}(\alpha)\,\frac{\exp[-\alpha r]}{r}\,\mathrm{d}\alpha\,,$$

where P is a Majorana exchange operator. There is no difficulty to extend our treatment to this case: to construct Γ_{d} and Γ_{e} one has to progress sym-

metrically with respect to $\cos\theta$. The important point is that obviously $\Gamma_e(\alpha)=0$ for $\alpha<2M$, and $\Gamma_d(\alpha)$ is given by the discontinuity of T_a *alone* for $\alpha<2M$. But since T_a is a purely elastic amplitude, from the above argument $\Gamma_d(\alpha)$ has to be purely real in the range $\alpha=\mu$, $\alpha=2M$. Hence the range (defined by the behaviour $\exp[-2Mr]$) of the imaginary potential cannot exceed $1/2M$.

One could provide many other applications of these fixed energy properties, in particular the study of the deuteron photodisintegration matrix element which has already been mentioned. Some people are trying to make an explicit calculation of the discontinuity for a Yukawa potential to see from what point the oscillations predicted by Regge start.

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A Self-Consistent Field Theory of Quantum Electrodynamics (*).

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1. – Introduction.

Present-day quantum electrodynamics attempts to unify the dynamical properties of a charged particle system with quantum mechanics. The former is assumed to follow, in strict accord with Einstein's principle of relativity, the Maxwell-Lorentz field theory. The latter depends on the association of a probability amplitude with the trajectory of each of the constituent charged particles. The probabilistic interpretation is inherent in the measuring process,

(*) A preliminary account of this paper was given at the Berkeley and New York meetings of the American Physical Society, *Bull. Amer. Phys. Soc.*, **5**, 505 (1960); **6**, 60 (1961).

(**) Part of this work was carried out while the authors were at the Department of Physical Science, San Jose State College, San Jose, California.

as expounded by Bohr's complementarity principle. The theory is therefore non-deterministic.

The apparent incompatibility between Einstein's deterministic, *completely field theoretical* approach and the non-deterministic approach of quantum mechanics has been the subject of much debate ever since the initial successes of quantum mechanics (1). Although this incompatibility is unimportant as far as the mathematical description of non-relativistic phenomena is concerned, it is indeed all-important in the exact (relativistic) description of quantum electrodynamics.

The deterministic approach of EINSTEIN has not been taken seriously by the majority of physicists for the two important reasons that non-relativistic quantum mechanics has had a great amount of success in its predictions and also because of the lack of an alternative deterministic theory which would duplicate the predictions of the properties of atomic and other elementary particle systems.

In order to explain high energy phenomena, it becomes necessary to extend the non-relativistic theory of quantum mechanics into the relativistic domain if relativistic covariance is to be taken as a principle underlying the description of all physical phenomena. The development of present-day quantum electrodynamics represents the construction of a field theory which attempts to incorporate the non-deterministic approach of quantum mechanics with Einstein's principle of relativity. A compilation of many of the important papers, representing the development of present-day quantum electrodynamics is given by SCHWINGER (2). Also, FOLDY (3) discusses the compatibility of relativistic covariance with the standard quantum-mechanical theory.

To display some of the successes of the present-day theory, one need but mention the calculations of the Lamb shift and the anomalous magnetic moment of the electron (4). On the other hand, the theory does suffer from some undesirable features. First, there are divergences which appear and must be removed by additional physical interpretation and further mathematical techniques in order to predict the measured properties of the system. Second, the conventional formalism does not give results in *closed form*. All of its predictions are inherently contained in expansions which represent the perturbing influence of the interactions within the system of particles on *free-particle fields*. Coupled with these objections is the question of the consistency of the present formulation of quantum electrodynamics.

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(1) See, for example, the article by N. BOHR and the reply by A. EINSTEIN in: *Albert Einstein Philosopher-Scientist* (Evanston, Ill., 1949), pp. 201, 666.

(2) J. SCHWINGER: *Quantum Electrodynamics* (New York, 1958).

(3) L. L. FOLDY: *Phys. Rev.*, **122**, 275 (1961).

(4) For a survey of these calculations see A. PETERMANN: *Fortschr. Phys.*, **6**, 505 (1958).

These difficulties in the present-day theory have their origin either in the mathematical formulation or in the conceptual framework. A great deal of work has been done on the exploration of the mathematical formulation in attempts to obtain a more consistent theory ([5]). It is our purpose in this work to present the results of a study that took as its starting point a reinvestigation of the conceptual framework of quantum electrodynamics. The approach that we take, similar to that of EINSTEIN, attempts to describe natural phenomena in terms of a completely covariant, non-singular and deterministic field theory.

An implication of the deterministic nature of the theory is that the solutions of the field equations provide a complete description of observable phenomena. This requires that the theory must necessarily be self-consistent. By self-consistency we mean that the system is closed (*i.e.*, the measuring apparatus as well as that which is observed is included in the description).

In the section which follows, the theoretical approach to our theory will be discussed, with emphasis given to our reinterpretation of the Dirac and Maxwell field variables. Section 3 then presents the Maxwell field equations in the two-component form developed by the authors ([6]). In Section 4 we develop a self-consistent field theory from a Lagrangian formalism. The Lagrangian density is expressed in terms of the bispinor Dirac field variables and the spinor variables of the Maxwell field. From the variational procedure we recover the Maxwell equations (in their spinor form) and a set of coupled non-linear Dirac equations.

The remaining sections in the paper apply the field equations to special cases. In Section 5 a result is derived for equivalent particles that is interpretable in terms of the Pauli exclusion principle. Section 6 is then devoted to the determination of the properties associated with a particular state of positronium. An *exact* solution is obtained for the coupled non-linear Dirac equations in the special case when the state of motion of the electron and positron are the same. It is found that this *bound* state of the particle-antiparticle pair exhibits all of the conserved properties that are normally associated with the annihilation of a pair and their simultaneous creation of a pair

([5]) One such approach of current interest attempts to provide a description of interacting particles without the use of « field » at all. Instead, physical events are described in terms of the scattering matrix that connects the state function of the particles in the remote past to the state function in the distant future, both of these times corresponding to the limit of no interaction. This phenomenological approach was originally introduced by HEISENBERG (*Zeits. f. Phys.*, **120**, 513, 673 (1943)) and most recently developed by LEHMANN, SYMANZIK and ZIMMERMANN (*Nuovo Cimento*, **1**, 205 (1955); **6**, 319 (1957)) and others. For a recent compilation of papers on this approach see L. KLEIN: *Dispersion Relations and the Abstract Approach to Field Theory* (New York, 1961).

([6]) S. L. SCHWEBEL and M. SACHS: *Bull. Amer. Phys. Soc.*, **5**, 505 (1960). Also, manuscript in preparation.

of oppositely circularly polarized photons. Thus, in our theory the pair is not truly annihilated, but rather has a ground state with associated properties which make it appear so. It is this state of positronium (or any particle-antiparticle pair) which replaces the vacuum state in the conventional theory. It is also shown how the electromagnetic coupling to a third charged particle can give rise to the phenomenon interpreted as pair production.

In Section **7** we show how the hydrogen spectrum (including the Lamb shift) emerges from our field equations. The Lamb shift $\Delta(n'S_{\frac{1}{2}} - n'P_{\frac{1}{2}})$ in hydrogen is calculated for $n' = 2, 3, 4$ and shows very good agreement with experiment and the conventional calculations, although no divergences appear at any stage in our calculation.

Finally, Section **8** examines the limitations of the present work and discusses its extension to other types of interactions.

2. – Theoretical approach.

The theoretical approach to the development of the covariant self-consistent field theory to be presented here is based on the following fundamental postulate:

> *The laws of nature must be described in terms of field variables which may be associated only with elementary interactions.*

It follows that the most elementary physical system is a two-body system because an interaction takes place between at least two physical particles. Thus we replace the *absolute* concept of the *elementary particle* by the *operational* concept of the *elementary interaction.*

The dynamical behavior of the system is described in terms of the solutions of the quantum mechanical equations. The solutions of these equations are then related to the interaction of the constituent particles of the system at the space-time point $\boldsymbol{x}$. Thus, $\boldsymbol{x}$ does not locate any individual particle; rather, if defines a point at which all of the particles interact. As a result of our requirement that the interaction rather than the free particle be considered elementary, the Dirac field variables, for the simplest system must be determined from a set of two coupled non-linear differential equations rather than the single (linear) free particle equation. As the number of particles in the system increases, the number of coupled Dirac field equations must correspondingly increase. Since however, the solutions of each of the n equations are not independent of each other, we postulate that a single interaction field function $\Psi(\psi^{(1)}, ..., \psi^{(n)})$ exists, whose absolute square is a measure of the *weighting* of the elementary interaction at the space-time point $\boldsymbol{x}$. We thus

require a conservation law (*)

$$\partial_\mu(i\overline{\Psi}\gamma_\mu\Psi) = 0 \tag{2.1}$$

to hold and interpret it as conservation of interaction. The normalization of Ψ, giving

$$\int \overline{\Psi}\gamma_0\Psi \, \mathrm{d}^3x = 1 \,, \tag{2.2}$$

is thereby interpreted to mean that the integrand weights the interaction between the constituent particles of the system throughout space. Such a result would then imply that as time progresses, particles could not be created from a vacuum, nor be annihilated with the subsequent creation of non-interacting particles. Such cases alter the value of (2.2). The first increases the total interaction while the second decreases it. Thus, our theory must predict the experimental results which are interpreted as pair creation and annihilation without actually creating or annihilating physical particles. This result is indeed obtained when we consider a particular state of the bound particle-antiparticle pair (Section 6).

In essence, our interpretation of the solutions of the Dirac field equations replaces the description of an n-particle system, in terms of space-time coordinates tied to each particle, by a four-dimensional space-time with an associated interaction field.

Once $\psi^{(1)}(\boldsymbol{x})$ and $\psi^{(2)}(\boldsymbol{x})$ of the two particle system (or $\psi^{(1)}(\boldsymbol{x})$, ..., $\psi^{(n)}(\boldsymbol{x})$) of the n-particle system) are determined, the conserved properties of the system may be obtained from the invariance properties of the Lagrangian density, in accordance with Noether's theorem. Thus, to determine the constants of the system it is only necessary to have the solutions $\psi^{(1)}$, $\psi^{(2)}$ and does not require the form of the Ψ field variable as a function of $\psi^{(1)}$ and $\psi^{(2)}$. On the other hand, a knowledge of the functional form of the Ψ field gives us information pertaining to the actual weighting of the interaction throughout space-

(*) The notation used is

$$\partial\left(\frac{\partial}{\partial \boldsymbol{x}}\right) \quad \{\partial_0 \quad i\partial_l ; \partial_k\}\,, \qquad \boldsymbol{x} \quad \{x_0 \quad il ; x_k\}\,,$$

$$\boldsymbol{\gamma} = \{\gamma_0 ; \gamma_k\}\,, \qquad \gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma_k = -i\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$$

$$\overline{\psi} = \psi^\dagger\gamma_0\,, \qquad \psi^\dagger_{ij} = \psi^*_{ji}\,, \qquad k = 1, 2, 3,$$

and σ_k are the usual Pauli matrices.

Also, we take $\hbar = c = 1$ throughout.

time. The derivation of this field function, for the case of equivalent particles in the same state of motion (Section 5), leads to the Pauli principle. It is also used (Section 6) to describe the dynamics of a particular state of a particle-antiparticle pair and to identify it with the standard interpretation of the annihilation of a pair with the simultaneous creation of a pair of oppositely circularly polarized photons.

Not only the interaction field Ψ, but also the individual Dirac functions $\psi^{(i)}$ satisfy the continuity condition (2.1). This is, in general, a consequence of the invariance of our Lagrangian (Section 4) with respect to gauge transformations of the first kind (*i.e.*, a phase change in $\psi^{(i)}$). We interpret

$$\partial_\mu(i\overline{\psi^{(i)}}\gamma_\mu\psi^{(i)}) = 0 \tag{2.3}$$

within the framework of our theory, to mean that each of the sources of the total interaction is conserved. This, once again, supports our earlier statement that within our theory particles cannot truly be created or annihilated. In view of (2.3), we then interpret the individual $\psi^{(i)}$ field variables to describe the kinematic behavior of the constituent interacting particles through the velocity field $\overline{\psi^{(i)}}\gamma_\mu\psi^{(i)}$.

Thus far, the discussion has developed our interpretation of the Dirac field as a consequence of the basic postulate of the elementary interaction. As a next step in this discussion, we must reinterpret the Maxwell field variables in a way which differs from that of present-day quantum electrodynamics.

In view of our assertion that the interaction rather than the free particle is elementary, we can accept the statement of the existence of a particle only in so far as it can influence other particles in an observable way. In particular, the Maxwell field variables are another way of representing a particle in an elementary interaction. Then, Maxwell's equations give us a covariant means of determining this representation of the particle in terms of these field variables. Based on this interpretation, we accept only the particular solutions of Maxwell's equations as physically meaningful. For if there are no sources then there can be no field variables and vice-versa. It then follows that the concepts of a source-free radiation field and of a photon, as an elementary particle, must be abandoned. These are replaced, in the present theory, by the concept of the propagation of interactions between particles. Such a concept was discussed by TETRODE (7) and by WHEELER and FEYNMAN (8).

According to our interpretation of the Maxwell field equations, the field variables are just another way to representing the particle's behavior. Thus in an elementary interaction, the influence of one particle on another can be

(7) H. TETRODE: *Zeits. f. Phys.*, **10**, 317 (1922).

(8) J. A. WHEELER and R. P. FEYNMAN: *Rev. Mod. Phys.*, **17**, 157 (1945).

described either in terms of the variables associated with its motion or in terms of the related Maxwell field variables, *but not both*. Consequently, such a concept as that of a particle acting on itself has no place in the present theory. (Nevertheless, when describing an elementary interaction we obtain formulas which are generalizations of those found in conventional electromagnetic theory [6]. If we relax the distinction between the particle and its representation by field variables, then the usual expressions are obtained.)

The present theory is deterministic. Let us consider the nature of an elementary interaction. No preference is given to either of the constituent particles; either may be the « observer » and the other the « observed ». It is impossible to separate the observer from the observed, and therefore a measurement on the observed (which is the function of an observer) is an integral part of the elementary interaction and completes its description. This requires a formulation of the theory which is symmetric with respect to an interchange of the field variables representing the observer with those of the observed. It should be emphasized that the completeness of the description results from the union of quantum mechanics with Einstein's theory of relativity.

Within the framework of our theory, quantum mechanics is interpreted as a statistical theory of elementary interactions. This is because the measuring process contemplated by the quantum theory is made up of a distribution of elementary interactions, and not of elementary particle trajectories. It is for this reason that we have not found it necessary to describe our field theory in terms of second quantization.

In essence, our theory returns to an Einstein-Maxwell concept of a field theory by replacing the particle with the interaction as an elementary concept.

3. – Maxwell's equations in spinor form.

It is shown by the authors [6] that the four conventional Maxwell equations may be expressed in terms of a pair of uncoupled two-component spinor equations. If we make the following identification with the conventional field variables

$$(3.1a) \qquad \varphi_1 = \begin{pmatrix} (H_3 - iE_3) \\ (H_1 - iE_1) + i(H_2 - iE_2) \end{pmatrix}, \qquad \varphi_2 = \begin{pmatrix} -(H_1 - iE_1) + i(H_2 - iE_2) \\ (H_3 - iE_3) \end{pmatrix},$$

$$(3.1b) \qquad \Upsilon_1 = 4\pi i \begin{pmatrix} -\varrho + j_3 \\ j_1 + ij_2 \end{pmatrix}, \qquad \Upsilon_2 = 4\pi i \begin{pmatrix} -j_1 + ij_2 \\ \varrho + j_3 \end{pmatrix},$$

then the four Maxwell equations can be expressed in the form of two uncoupled spinor equations

$$(3.2) \qquad \sigma_\mu \partial_\mu \varphi_\alpha = \Upsilon_\alpha \qquad (\alpha = 1, 2),$$

where $\sigma_0 = iI$ (I is the unit 2×2 matrix) and σ_k are the usual Pauli matrices.

The covariance of this form of the Maxwell field equations is discussed in the Appendix of reference ([6]).

We now specify that there is a set of Maxwell field eq. (3.2) for each constituent particle of a system. Thus, the p-th particle of an n-particle system is described in four-space by the source terms $\Upsilon_\lambda^{(p)}(\boldsymbol{x})$ or the associated solutions $\varphi_\lambda^{(p)}(\boldsymbol{x})$ of the field eq. (3.2). It is emphasized that $\boldsymbol{x}$ is not a function of the p-th particle. The four co-ordinates (x_0, x_1, x_2, x_3) are to be regarded as one does in a purely field theoretical description, as a set of four parameters which define a metric and a corresponding space which maps the field variables $\Upsilon_\lambda(\boldsymbol{x})$ and $\varphi_\lambda(\boldsymbol{x})$. It should also be emphasized that the conventional field variables ($\boldsymbol{E}$, $\boldsymbol{H}$, j_μ) are introduced in (3.1) only for the purposes of identification with our field variables in one particular Lorentz-frame. The variables of the field theory which is developed here are φ_1, φ_2 and the Dirac variable ψ (identified below with Υ_α). The transformation properties of the Maxwell field variables which leave the field eq. (3.2) Lorentz-covariant are therefore those of two-component (first-rank) spinors (see the Appendix in ref. ([6])). Eq. (3.2) may then be regarded as a factorization of the vector (spin-one) field equations. No inconsistency is introduced by this factorization since we have abandoned the concept of the photon as an elementary particle.

Finally, we must identify the source terms Υ_α in (3.2) with the Dirac field variables. To do this we note the identification of Υ_α with the components of j_μ in (3.1). The usual identification (for the p-th particle) is then made as follows:

$$j_\mu^{(p)} = ie^{(p)}\overline{\psi^{(p)}}\gamma_\mu\psi^{(p)}\,, \tag{3.3}$$

where $e^{(p)}$ may be regarded here as a proportionality factor. With (3.3), the Maxwell field eq. (3.2) take the form (for the p-th particle),

$$\sigma_\mu\partial_\mu\varphi_\alpha^{(p)}(\boldsymbol{x}) = e^{(p)}\overline{\psi^{(p)}}(\boldsymbol{x})\Gamma_\alpha\psi^{(p)}(\boldsymbol{x})\,, \tag{3.4}$$

where

$$\psi^{(p)}\Gamma_1\psi^{(p)} = 4\pi i\begin{pmatrix}\psi^{(p)}(-\gamma_0+i\gamma_3)\psi^{(p)}\\ \psi^{(p)}(i\gamma_1-\gamma_2)\psi^{(p)}\end{pmatrix} = \begin{pmatrix}\psi^{(p)}\Gamma_1(1)\psi^{(p)}\\ \overline{\psi^{(p)}}\Gamma_1(2)\psi^{(p)}\end{pmatrix}, \tag{3.5a}$$

$$\psi^{(p)}\Gamma_2\psi^{(p)} = 4\pi i\begin{pmatrix}\psi^{(p)}(-i\gamma_1-\gamma_2)\psi^{(p)}\\ \overline{\psi^{(p)}}(\gamma_0+i\gamma_3)\psi^{(p)}\end{pmatrix} \equiv \begin{pmatrix}\overline{\psi^{(p)}}\Gamma_2(1)\psi^{(p)}\\ \overline{\psi^{(p)}}\Gamma_2(2)\psi^{(p)}\end{pmatrix}. \tag{3.5b}$$

Finally, a comment should be made regarding the solutions of (3.4). In view of our interpretation of these equations (discussed in the preceding section) we accept only the particular solutions of (3.4) as physically meaningful. Thus, if the right-hand side of (3.4) vanishes, we must take the solution φ_λ

to be identically zero. Similarly, if the solutions are zero, the source on the right-hand side of (3.4) is taken to be identically zero.

In view of the formulation of the Maxwell field equations presented here, we are now in a position to construct a covariant self-consistent field theory in terms of the coupling of spinor fields.

4. – A self-consistent field theory.

It is our purpose, in this section, to construct a Lagrangian density function which leads, in accordance with the principle of least action, to the relativistically covariant self-consistent field equations for a system of two charged spin one-half particles. We require at the ouset that such a Lagrangian satisfy the requirements of Lorentz and gauge invariance, of hermiticity and be symmetric with respect to the interchange of the field variables associated with each particle. From its variation we require a recovery of the electromagnetic field eq. (3.4) and a generalized Dirac equation.

4·1. *The principle of least action.* – Since we are dealing with a *c*-number field theory, the standard application of the variational principle to classical fields is valid. We now specify that the two particle Lagrangian density (*i.e.*, the most elementary form of the Lagrangian) depends explicitly on the Maxwell field spinors $q_\lambda^{(1)}$, $q_\alpha^{(1)\dagger}$, $q_\lambda^{(2)}$, $q_\alpha^{(2)\dagger}$, on the Dirac field bispinors $\psi^{(1)}$, $\psi^{(\bar{1})}$, $\psi^{(2)}$, $\overline{\psi^{(2)}}$ and on their respective first derivatives. Thus we shall consider as fundamental the spinors $q_\alpha^{(1)}$, $q_\lambda^{(2)}$ and $\psi^{(1)}\psi^{(2)}$ and shall investigate the equations of motion of the system in terms of these functions.

In the limiting case in which the electromagnetic interactions become small in comparison with the rest energy of the system, the terms contributing to the Dirac equation are obtained from the following portion of the Lagrangian density (*):

$$\mathscr{L}_D = \sum_{p=1}^{2} \{\psi^{(p)}(\gamma_\mu \partial_\mu + m^{(p)})\psi^{(p)} + (-\partial_\mu \psi^{(p)}\gamma_\mu + m^{(p)}\psi^{(p)})\psi^{(p)}\} . \tag{4.4}$$

The portion of the Lagrangian density which gives rise to the Maxwell field eq. is expressed analogously to (4.4) and (3.4) has the following general form:

$$\mathscr{L}_M = ig_M \sum_{p \neq q}^{2} \sum_{\lambda=1}^{2} a_\alpha \varphi_\alpha^{(p)\dagger}(\sigma_\mu \partial_\mu q_\alpha^{(q)} - 2e^{(q)}\psi^{(q)} \Gamma_\alpha \psi^{(q)}) + \text{h. c.} , \tag{4.5}$$

(*) The arbitrary constant that multiplies the righ-hand side of (4.4) must be, (to agree with experiment) $\hbar c$. This is where $\hbar$ enters the theory.

where g_M (a real constant with the dimensions of length) represents the one extra fundamental constant of our theory and where a_1 and a_2 are dimensionless numbers, to be determined below. It should be noted that $\mathscr{L}_M$ contains an interaction term which couples the Maxwell field variables q_λ to the Dirac field variables through the source term $\overline{\psi}\Gamma_\lambda\psi$. Note that our restriction $p \neq q$ follows from our discussion in Section 2 which prohibits a particle to act on itself.

The remaining part of the interaction is then obtained from the part of the Lagrangian which couples the velocity fields of the particles. It has the form:

$$\mathscr{L}_L = -2\sum_{p\neq q=1}^{2} e^{(p)}e^{(q)}\overline{\psi^{(p)}}\gamma_\nu\psi^{(p)}(\boldsymbol{x})\int\overline{\psi^{(q)}}\gamma_\nu\psi^{(q)}(\boldsymbol{x}')\,S(\boldsymbol{x}-\boldsymbol{x}')\,\mathrm{d}^4x' + \tag{4.6}$$

$$+\,i\,\frac{\xi}{2}\,e^{(1)}e^{(2)}\varepsilon_{12}\varepsilon_{\mu\nu\lambda\varrho}\overline{\psi^{(1)}}\gamma_\nu\psi^{(1)}(\boldsymbol{x})\int\overline{\psi^{(2)}}\gamma_\lambda\psi^{(2)}(\boldsymbol{x}')\,\mathrm{d}^4x'\int_{\mathrm{I}}\partial_\varrho S(\boldsymbol{x}-\boldsymbol{x}')\,\mathrm{d}x_\mu\,,$$

where

$$S(\boldsymbol{x}-\boldsymbol{x}') = \frac{1}{(2\pi)^4}\int\frac{\mathrm{d}^4k}{k_\nu^2}\exp\left[ik_\mu(x_\mu - x'_\mu)\right] = \tag{4.7}$$

$$= \frac{1}{8\pi|\boldsymbol{r}-\boldsymbol{r}'|}\left\{\delta\left[(t-t')-|\boldsymbol{r}-\boldsymbol{r}'|\right]+\delta\left[(t-t')+|\boldsymbol{r}-\boldsymbol{r}'|\right]\right\}$$

is the Green's function for D'Alembert's equation, *i.e.*,

$$\Box S(\boldsymbol{x}-\boldsymbol{x}') = -\,\delta(\boldsymbol{r}-\boldsymbol{r}')\,\delta(t-t') \qquad (\Box \equiv \nabla^2 - \partial_t^2). \tag{4.8}$$

The symbol $\varepsilon_{\mu\nu\lambda\varrho}$ is the antisymmetric unit four-tensor in $(\mu\nu\lambda\varrho)$ and ε_{12} acts as an odd permutation operator on the particle indices (1) and (2) of the ψ functions. The subscript I on the last integral in eq. (4.6) indicates an indefinite integral and the symbol d^4 indicates an integration over all four-space.

The first part of $\mathscr{L}_L$ is equivalent to the scalar interaction term

$$\sum_{p\neq q=1}^{2} e^{(p)}\psi^{(p)}\gamma_\mu\psi^{(p)}A_\mu^{(q)}$$

which appears in the conventional theory, except for the particle aspects of this theory, indicated by the superscripts p and q. Similarly, the second part of $\mathscr{L}_L$ is equivalent (except for the p, q notation) to the pseudoscalar interaction term

$$\sum_{p\neq q=1}^{2} e^{(p)}\psi^{(p)}\gamma_\mu\psi^{(p)}B_\mu^{(q)}$$

which appears in the generalized Dirac equation (9). The replacement of $S(\boldsymbol{x}-\boldsymbol{x}')$ in eq. (4.6) by the right-hand side of eq. (4.7) describes the coupling of velocity fields in terms of the advanced and retarded solutions of the field equations.

Although there exist many invariant interactions expressible in terms of the field variables of our theory, we have chosen the Lagrangian density $\mathscr{L}_L$ because it has a classical counterpart in the generalized theory.

Summarizing, the total action function whose vanishing variation gives the behavior of a system of two charged particles is

$$I=\int_{t'}^{t}\!\!\int(\mathscr{L}_D+\mathscr{L}_M+\mathscr{L}_L)\,\mathrm{d}^3x\,\mathrm{d}t=\int_{t'}^{t}\!\!\int\mathscr{L}\,\mathrm{d}^3x\,\mathrm{d}t\,, \tag{4.9}$$

where $\mathscr{L}_D$, $\mathscr{L}_M$, and $\mathscr{L}_L$ are given by eqs. (4.4), (4.5) and (4.6), respectively. The extension to many particles is then straightforward.

4·2. *The field equations.* – The derivation of the field equations follows from the principle of least action. The vanishing of the variational derivative of the Lagrangian density with respect to the spinor fields $(\varphi_\alpha, \varphi_\lambda^\dagger)$ and $(\psi, \bar{\psi})$ give respectively the Maxwell eq. (3.4) and the Dirac equations.

Taking the variational derivative of $\mathscr{L}$ with respect to the independent Dirac field variable $\psi^{(p)}$ (with $p=1$ and 2), we obtain the following coupled field equations:

$$\hat{O}(1,2)\psi^{(1)}=0\,, \tag{4.10a}$$

$$\hat{O}(2,1)\psi^{(2)}=0\,, \tag{4.10b}$$

where the operator $\hat{O}(1,2)$ has the form

$$\begin{aligned}\hat{O}(1,2)=\Big\{\gamma_\nu\Big[\partial_\nu-e^{(1)}e^{(2)}\Big(\int\overline{\psi^{(2)}}\gamma_\nu\psi^{(2)}S(\boldsymbol{x}-\boldsymbol{x}')\,\mathrm{d}^4x'-\\ -i\,\frac{\xi}{4}\,\varepsilon_{\mu\nu\lambda\varrho}\int\overline{\psi^{(2)}}\gamma_\lambda\psi^{(2)}\,\mathrm{d}^4x'\int\partial_\varrho S(\boldsymbol{x}-\boldsymbol{x}')\,\mathrm{d}x_\mu\Big)\Big]-\\ -ig_Me^{(1)}\sum_{\alpha=1}^{2}a_\alpha[\varphi_\alpha^{(2)\dagger}\cdot\Gamma_\alpha-(\gamma_0\Gamma_\alpha^\dagger\gamma_0)\cdot\varphi_\alpha^{(2)}]+m^{(1)}\Big\}\,,\end{aligned} \tag{4.11}$$

and where

$$\varphi_\alpha^\dagger\cdot\Gamma_\alpha\equiv\varphi_\alpha^*(1)\,\Gamma_\alpha(1)+\varphi_\lambda^*(2)\,\Gamma_\lambda(2)\,, \tag{4.12}$$

$(\Gamma_\lambda(1),\ \Gamma_\lambda(2))$ are the $4\cdot4$ matrices defined in eqs. (3.5*a* and *b*) and $(\varphi_\lambda(1),\ \varphi_\lambda(2))$ are the components of the Maxwell field spinor φ_α.

(9) M. Sachs: *Ann. of Phys.*, **6**, 244 (1959).

The coupled set of Dirac field eq. (4.10), together with the Maxwell field eq. (3.4) and the interpretation of the field variables provided by the present theory, constitutes a completely self-consistent and relativistically covariant field theory of the elementary electrodynamical interaction.

It should be noted that the non-linearity of the Dirac field equations is a consequence of the requirement that its solutions represent a complete description of the elementary interaction. In contrast, the Maxwell field equations are linear because they are descriptions of individual sources.

4·3. *The continuity condition.* – Because of the hermitian property of the Hamiltonian operators associated with eq. (4.10), it follows in the usual manner that the following conservation law is valid (*):

$$\partial_\mu(i\psi^{(p)}\gamma_\mu\psi^{(p)}) = 0 \qquad (p=1, 2). \tag{4.13}$$

However, in view of our operational description of elementary interactions, the four-vectors $\overline{\psi^{(1)}}\gamma_\mu\psi^{(1)}$ and $\overline{\psi^{(2)}}\gamma_\mu\psi^{(2)}$ may not by themselves be interpreted as functions related to the observable measurements. As was pointed out earlier, the function which provides a complete description of the system must depend on both $\psi^{(1)}$ and $\psi^{(2)}$ and also satisfy the continuity equation

$$\partial_\mu\{i\overline{\Psi}(\psi^{(1)}, \psi^{(2)})\gamma_\mu\Psi\psi^{(1)}, \psi^{(2)})\} = 0\,. \tag{4.14}$$

It is the time component $i\overline{\Psi}\gamma_0\Psi(\boldsymbol{x})$ that weights the interaction at the space-time point $\boldsymbol{x}$. The condition (4.13) follows from the requirement that the source of the particle force field be conserved. We now wish to consider some of the properties of the field variable Ψ as a function of $\psi^{(1)}$ and $\psi^{(2)}$ for a special case.

5. – The Pauli exclusion principle.

Consider, once again, the form of the generalized Dirac equations given by (4.10). Multiplying eq. (4.10*a*) on the left by $\overline{\psi^{(2)}}$ and the hermitian conjugate of eq. (4.10*b*) on the right by $\psi^{(1)}$, subtracting these two equations and then repeating the operation with (1) and (2) interchanged gives us the following result

$$\begin{aligned}\partial_\nu(\overline{\psi^{(2)}}\gamma_\nu\psi^{(1)} + \overline{\psi^{(1)}}\gamma_\nu\psi^{(2)}) = &- e^{(1)}e^{(2)}(\overline{\psi^{(1)}}\gamma_\nu\psi^{(2)} - \overline{\psi^{(2)}}\gamma_\nu\psi^{(1)})P_\nu + \\ &- ig_M\sum_{\alpha=1}^{2}\{Q_\alpha(\overline{\psi^{(2)}}\Gamma_\alpha\psi^{(1)} - \overline{\psi^{(1)}}\Gamma_\alpha\psi^{(2)}) - \text{h. c.}\} + (m^{(2)} - m^{(1)})(\overline{\psi^{(2)}}\psi^{(1)} - \overline{\psi^{(1)}}\psi^{(2)})\,,\end{aligned} \tag{5.1}$$

(*) This, of course, also follows from the invariance of $\mathscr{L}$ with respect to gauge transformations of the first kind.

where

$$P_\nu = \Big\{ \int (\overline{\psi^{(2)}}\gamma_\nu\psi^{(2)} - \overline{\psi^{(1)}}\gamma_\nu\psi^{(1)})\, S(\boldsymbol{x}-\boldsymbol{x}')\, \mathrm{d}^4x' - \tag{5.2}$$

$$- i\frac{\xi}{4}\varepsilon_{\mu\nu\lambda\varrho}\int (\overline{\psi^{(2)}}\gamma_\lambda\psi^{(2)} - \overline{\psi^{(1)}}\gamma_\lambda\psi^{(1)})\, \mathrm{d}^4x' \int_\Gamma \partial_\varrho\, S(\boldsymbol{x}-\boldsymbol{x}')\, \mathrm{d}x_\mu \Big\} ,$$

$$Q_\alpha = a_\alpha\{e^{(1)}\varphi_\alpha^{(2)\dagger} - e^{(2)}\varphi_\alpha^{(1)\dagger}\} . \tag{5.3}$$

According to the Maxwell eq. (3.4), we find that

$$\sigma_\mu\partial_\mu(e^{(1}\varphi_\alpha^{(2)} - e^{(2)}\varphi_\alpha^{(1)}) = e^{(1)}e^{(2)}(\overline{\psi^{(2)}}\Gamma_\alpha\psi^{(2)} - \overline{\psi^{(1)}}\Gamma_\alpha\psi^{(1)}) . \tag{5.4}$$

Therefore, if the state of motion of each of the particles (*i.e.*, $\overline{\psi}\Gamma_\alpha\psi$) is the same, the right-hand side of eq. (5.4) vanishes. This implies that

$$\psi^{(1)}\gamma_\nu\psi^{(1)}(\boldsymbol{x}) = \psi^{(2)}\gamma_\nu\psi^{(2)}(\boldsymbol{x})$$

for all ν, so that both P_ν and Q_α vanish.

Eq. (5.4) is just of the form of the Maxwell eq. (3.4). The reason that Q_α vanishes is a consequence of our interpretation of these equations. It is only when sources are present that the field variables are non-zero.

If we further assume that the two particles have the same inertial mass, eq. (5.1) reduces to

$$\partial_\nu(\psi^{(2)}\gamma_\nu\psi^{(1)} + \psi^{(1)}\gamma_\nu\psi^{(2)}) = 0 . \tag{5.5}$$

Combining eqs. (5.5) and (4.13) and also taking into account the symmetry with respect to an interchange between (1) and (2) imposed by the theory, we obtain the following result:

$$\partial_\nu[(\psi^{(1)}\gamma_\nu\psi^{(1)} + \overline{\psi^{(2)}}\gamma_\nu\psi^{(2)}) \pm (\overline{\psi^{(2)}}\gamma_\nu\psi^{(1)} + \psi^{(1)}\gamma_\nu\psi^{(2)})] = \partial_\nu(\overline{\Psi}\gamma_\nu\Psi) = 0 , \tag{5.6}$$

where

$$\Psi(\boldsymbol{x}) = (\psi^{(1)} \pm \psi^{(2)})(\boldsymbol{x}) . \tag{5.7}$$

The state function $\Psi(\boldsymbol{x})$ provides a complete description of the elementary interaction between two charged particles which have equal masses and are in the same state of motion.

In the conventional linear theory, it would suffice to note that when the masses of each particle and their interactions with a background force field are the same, $\psi^{(1)}$ and $\psi^{(2)}$ would be solutions of the same differential equation, so that the linear combination (5.7) would also be a solution. The present

non-linear theory differs, therefore, from the conventional theory in the necessity of specifying that the solution (5.7) is valid for a system of equal mass particles only if the state of motion of each of the particle is the same.

Summarizing thus far, we see that the function $(\psi^{(1)} \pm \psi^{(2)})(\boldsymbol{x})$ satisfies the continuity eq. (5.6). Asserting that the interaction field variable has a unique form (which follows from the specification of a complete description) it is clear that the solution must either be $(\psi^{(1)}+\psi^{(2)})$ or $(\psi^{(1)}-\psi^{(2)})$. The result obtained above applies to two possible cases. These are the system of equivalent particles, each in the same state of motion, and that of the particle and antiparticle, each in the same state of motion. The respective interactions are characterized by $+e^2$ and $-e^2$. To resolve which of the solutions applies, we consider the case of two equivalent particles. The determination of Ψ from this case must necessarily apply to the second case.

If the two particles are equivalent then

$$\psi^{(1)}(\boldsymbol{x}) \equiv \psi^{(2)}(\boldsymbol{x}) \tag{5.8}$$

and thus, either

$$\Psi = \psi^{(1)} + \psi^{(2)} = 2\psi \tag{5.9a}$$

or

$$\Psi = \psi^{(1)} - \psi^{(2)} = 0\,. \tag{5.9b}$$

The solution ψ (5.8) is determined from the Dirac equation

$$\hat{O}(\psi)\psi = 0$$

which has all of the mathematical consequences of a particle acting on itself. The acceptance of the non-zero result (5.9a) is then incompatible with the basic premise of our theory which rejects the notion of a particle acting on itself. On the other hand, the solution (5.9b) indicates that if we approach a situation in which the field equations describe a particle acting on itself, the interaction field automatically vanishes at all points in space-time. Thus, as a logical consequence of the fundamental premise of our theory and the assertion of the uniqueness of Ψ we choose the solution $(\psi^{(1)} - \psi^{(2)})$. This result then shows that the interaction between two identical particles, each in the same state of motion, is not an observable.

The foregoing result is, therefore, in accordance (mathematically) with the statement of the Pauli exclusion principle. It should be emphasized, however, that the physical statement of the Pauli principle and our result are quite different because of the different interpretation of the wave function. In the conventional theory, $\psi^{(1)}(\boldsymbol{x}_1)$ and $\psi^{(2)}(\boldsymbol{x}_2)$ are interpreted separately as proba-

bility amplitudes which locate the physical particles, while in this theory, the state function $(\psi^{(1)} - \psi^{(2)})(\boldsymbol{x})$ provides a complete description of the elementary interaction. It is the replacement of the description in terms of *locations* of particles by a description in terms of an interaction *field* which renders the present theory a field theory, in accordance with the fundamental postulates of Einstein's theory of relativity.

Since the mathematical result obtained here is equivalent to that of PAULI, the generalization to the many-electron system follows in the same way, and all consequences of the exclusion principle (such as atomic structure) are in complete agreement.

The special case of the particle-antiparticle system, in which the state of motion of both particles is the same, is shown in the succeeding section to be characterized by

$$\psi^{(1)}(\boldsymbol{x}) = -\psi^{(2)}(\boldsymbol{x}) .$$

This situation corresponds to the symmetric function which occurs in the conventional theory.

6. – Positronium and the pair annihilation-creation processes.

The particle-antiparticle system is the simplest one to treat, owing to the fact that the Dirac function for one particle can be obtained directly from that of the other particle by applying the Wigner operator ([10]) to the latter. Thus, if one of the coupled Dirac equations (describing the electron) has the form (4.10*a*),

$$\{\gamma_\mu \partial_\mu - e^- \mathscr{A}(e^+) + m\}\psi(e^-) = 0 \tag{6.1a}$$

the other equation (describing the positron) has the form

$$\{\gamma_\mu \partial_\mu - e^+ \mathscr{A}(e^-) + m\}\psi(e^+) = 0 , \tag{6.1b}$$

where

$$\psi(e^+) = \hat{W}\psi(e^-) , \qquad e^+ = -e^- = e > 0 , \tag{6.2}$$

$$\mathscr{A}(e) = e\left\{\gamma_\nu \int \bar{\psi}(e)\gamma_\nu \psi(e) S(\boldsymbol{x} - \boldsymbol{x}')\, \mathrm{d}^4x' - \right. \tag{6.3}$$

$$- i\frac{\xi}{4}\varepsilon_{\mu\nu\lambda\varrho}\gamma_\nu \int \bar{\psi}(e)\gamma_\lambda \psi(e)\, \mathrm{d}^4x' \int_{\mathrm{I}} \partial_\varrho S(\boldsymbol{x} - \boldsymbol{x}')\, \mathrm{d}x_\mu -$$

$$\left. - ig_M \sum_{\alpha=1}^{2} a_\alpha [\varphi_\alpha^\dagger(e)\cdot \Gamma_\alpha - \gamma_0 \Gamma_\alpha^\dagger \gamma_0 \cdot \varphi_\alpha(e)]\right\} ,$$

([10]) E. P. WIGNER: *Group Theory and its Application to Quantum Mechanics of Atomic Spectra* (New York, 1959), chap. 26.

and

$$\hat{W} = \gamma_2 \hat{K}_0 \equiv \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \hat{K}_0$$

is the Wigner operator. The operator $\hat{K}_0$ denotes the complex conjugation operation.

Equation (6.1*b*) (for the antiparticle) is obtained from eq. (6.1*a*) (for the particle) by applying the Wigner operator to the latter. This is precisely the companion eq. (4.10*b*) required by our theory for the particle-antiparticle system. The complete description of positronium is therefore provided by the self-consistent eq. (6.1) together with the Maxwell field eq. (3.4).

We now wish to solve this set of equations for the special case in which the state of motion of the particles is the same. The Maxwell field eq. (3.4) for the electron and positron are

$$\sigma_\nu \partial_\nu \varphi_\alpha(e^-) = -\, e\overline{\psi}_n(e^-) \Gamma_\alpha \psi_n(e^-) \,, \tag{6.4a}$$

$$\sigma_\nu \partial_\nu \varphi_\alpha(e^+) = e\overline{\psi}_{n'}(e^+) \Gamma_\alpha \psi_{n'}(e^+) \,, \tag{6.4b}$$

where the quantum numbers n and n', associated with the eigensolutions of the coupled Dirac eq. (6.1), specify different states of motion of the respective particles. If we now substitute $\hat{W}\psi_{n'}(e^-)$ for $\psi_{n'}(e^+)$ in eq. (6.4*b*), we have

$$\sigma_\nu \partial_\nu \varphi_\alpha(e^+) = e\overline{\psi}_{n'}(e^-) \Gamma_\alpha \psi_{n'}(e^-) \,. \tag{6.4c}$$

The right-hand side of eq. (6.4*c*) follows from eq. (6.2), since

$$\overline{\psi}_{n'}(e^+)\gamma_\mu \psi_{n'}(e^+) = \left(\psi^\dagger_{n'}(e^-)\right)^* \gamma_2 \gamma_0 \gamma_\mu \gamma_2 \psi^*_{n'}(e^-) = \tag{6.4d}$$

$$= -\left(\overline{\psi}_{n'}(e^-)\right)^* \gamma_2 \gamma_\mu \gamma_2 \psi^*_{n'}(e^-) = \overline{\psi}_{n'}(e^-)\gamma_\mu \psi_{n'}(e^-)$$

and Γ_α depends on linear combinations of the Dirac matrices.

It then follows that if the quantum numbers n and n' are the same, the sum of eqs. (6.4*a*) and (6.4*b*) is

$$\sigma_\nu \partial_\nu \varphi_\alpha = 0 \,, \tag{6.5}$$

where

$$\varphi_\alpha = \varphi_\alpha(e^-) + \varphi_\alpha(e^+) \tag{6.6}$$

is the total electromagnetic spinor force field variable. Thus, we see that in this particular case the total electromagnetic field *appears* as a source-free radiation field.

We now observe that when $n = n'$,

$$e^+ \mathscr{I}(e^-) = e^- \mathscr{I}(e^+) \,. \tag{6.7}$$

This result follows from using eq. (6.4*d*) in eq. (6.3). It follows that if $\psi(e^-)$ is a solution of the non-linear Dirac field eq. (6.1*a*), then $\psi(e^+) = \hat{W}\psi(e^-)$ is the solution of the coupled equation, and the desired interaction amplitude is, from considerations of the preceding section,

$$\Psi = (1 - \hat{W})\psi(e^-) \,. \tag{6.8}$$

We find below a solution for which $\hat{W}\psi = -\psi$ so that eq. (6.8) has the form

$$\Psi = 2\psi \tag{6.9}$$

which describes the ground state of the electron-positron pair. This will be referred to as the *annihilation state*, since this state of positronium will be shown to exhibit all of the properties usually associated with the annihilation of an electron-positron pair.

6·1. *A solution of the field equations.* – A solution of the non-linear field eq. (6.1*a*) is

$$\psi = \begin{pmatrix} \exp[-imt] \\ 0 \\ 0 \\ \exp[imt] \end{pmatrix} \tag{6.10}$$

if $a_1 = -a_2$ in the interaction term (eq. (6.3)). Thus, taking $a_1 = -a_2 = +1$, the unique form of this interaction term is determined for all other cases. That (6.10) is a solution is readily verified by substituting eq. (6.10) into eq. (6.1*a*). As we show in the Appendix, with the solution (6.10), the interaction term operating on ψ is

$$\mathscr{I}(e)\psi(e) = 0$$

while the interaction operator $\mathscr{I}$ *itself does not vanish.* On the basis of the present theory, this result was anticipated for the annihilation state of the particle-antiparticle pair.

Let us now consider the components of the four-vector

$$i\overline{\Psi}\gamma_\mu\Psi \qquad \left(\Psi = (8V)^{-\frac{1}{2}}\psi = 2\psi_{(\text{normalized})}\right)$$

where $(8V)^{-\frac{1}{2}}$ is the normalization coefficient for Ψ in a volume V.

According to eq. (4.14), we interpret the time component as the « interaction weighting function » for the particles to interact. From eq. (6.9) and (6.10) we find that

$$\overline{\Psi}\gamma_0\Psi = 1/V \tag{6.11a}$$

and that the space components are

$$\begin{aligned} i\overline{\Psi}\gamma_1\Psi &= \frac{1}{V}\cos 2mt\,, \\ i\Psi\gamma_2\Psi &= \frac{1}{V}\sin 2mt\,, \\ i\Psi\gamma_3\Psi &= 0\,. \end{aligned} \tag{6.11b}$$

Using eq. (6.10), the source terms (3.5) of the Maxwell field equations become

$$\begin{cases} e\overline{\psi}\Gamma_1\psi = -8\pi ie\begin{pmatrix} & 1 \\ -\exp[2imt] & \end{pmatrix}, \\ e\overline{\psi}\Gamma_2\psi = \quad 8\pi ie\begin{pmatrix} -\exp[-2imt] & \\ & 1 \end{pmatrix}. \end{cases} \tag{6.12}$$

The interaction velocity field, given by eq. (6.11*b*), indicates that the particles interact in a plane transverse to the 3-axis. The source terms (6.12) are described by two independent (right- and left-handed) circularly polarized particle currents. Thus, the dynamical properties of the particle-antiparticle pair (in the annihilation state) may be compared with the conventional description of the annihilation process in terms of the creation of two photons.

We now proceed to calculate the energy-momentum of this state of positronium.

6·2. *The energy-momentum of positronium in the annihilation state.* – The invariance properties of the Lagrangian density imply conservation laws ([11]). In particular, invariance under translations gives the conservation of the energy-momentum vector of the system.

([11]) See, for example, N. N. BOGOLIUBOV and D. V. SHIRKOV: *Introduction to the Theory of Quantized Fields* (New York, 1959), Sect. 2.

The variations of the space-time coordinates and the field variables are given, respectively, by the equations.

$$\delta x_\mu = X_{\mu\nu}\,\delta\omega_\nu ; \qquad (x_\mu \to x'_\mu = x_\mu + \delta x_\mu) , \tag{6.13a}$$

and

$$\delta A_\mu = \Pi_{\mu\nu}\,\delta\omega_\nu ; \qquad \big(A_\mu(\boldsymbol{x}) \to A'_\mu(\boldsymbol{x}') = A_\mu(\boldsymbol{x}) + \delta A_\mu(\boldsymbol{x})\big) , \tag{6.13b}$$

where the ν summation extends over the number of independent parameters ω_ν.

The vanishing of the variation of the action function under the variations (6.13) yields the conservation law

$$\partial_\mu \Theta_{\mu\nu} = 0 , \tag{6.14}$$

where

$$\Theta_{\mu\nu} = \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\beta)} (\partial_\varrho A_\beta X_{\varrho\gamma} - \Pi_{\beta\nu}) - \mathscr{L} X_{\mu\nu} . \tag{6.15}$$

The independent field variables A_β are the Maxwell variables $q_\alpha^{(p)}$, $q_\alpha^{(p)\dagger}$ and the Dirac field variables $\psi^{(p)}$ and $\overline{\psi^{(p)}}$, $(p, \alpha = 1, 2)$.

When the co-ordinates are subjected to an infinitesimal translation $x_\mu \to x'_\mu = x_\mu + \delta\omega_\mu$, from eq. (6.13$a$) we have

$$\begin{cases} X_{\mu\nu} = \delta_{\mu\nu} , \\ \Pi_{\beta\nu} = 0 . \end{cases} \tag{6.16}$$

Substituting eq. (6.16) into eq. (6.15), we obtain the energy-momentum tensor

$$T_{\mu\nu} = \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\beta)} \partial_\nu A_\beta - \mathscr{L}\delta_{\mu\nu} . \tag{6.17}$$

From eq. (6.14), the energy-momentum vector which is conserved is

$$P_\nu = \int T_{0\nu}\,\mathrm{d}\boldsymbol{r} . \tag{6.18}$$

It is shown in the Appendix that, for the source terms (6.12), the Maxwell field eqs. (3.4) have as their solutions

$$q_1^{(p)} = \frac{4\pi e^{(p)}}{m} \begin{pmatrix} 1 \\ \exp[2imt] \end{pmatrix} , \tag{6.19a}$$

$$q_2^{(p)} = \frac{4\pi e^{(p)}}{m} \begin{pmatrix} \exp[-2imt] \\ 1 \end{pmatrix} . \tag{6.19b}$$

With eqs. (6.10) and (6.19), P_ν is readily evaluated from (6.15) and is found to vanish, (*i.e.*, P_ν is a null-vector). Thus, we have shown that the solution (6.10) represents the ground state of positronium characterized by its energy-momentum vector being zero.

The binding of the two particles in this state exhibits properties which conventional theory would interpret as annihilation. However, as eqs. (6.19) indicate, positronium in the annihilation state can couple electromagnetically to another charged particle. With sufficient energy it is possible for the third particle to excite the system into a state which is interpreted conventionally as pair production. The result that the total energy associated with this (ground) state of positronium is zero means that the ground state of this atom is $(2m)$ below the energy associated with the « limiting » state of the two free particles. It is this (ground) state of the particle-antiparticle pair which in our theory replaces the vacuum of the conventional theory. This is because it is from this state that a pair is created and into this state that a pair is annihilated in the conventional theory.

7. – The hydrogen spectrum.

We now consider the field eqs. (4.10) applied to the electron-proton system and show how the hydrogen spectrum evolves from their solutions. Whereas the application of the theory to the cases considered thus far necessitated the utilization of the full non-linearity of the coupled field equations, the complexity of the application to hydrogen has led us to consider a linearization procedure as a first approximation. This is done below by allowing the proton mass to become infinitely large. It is our purpose in this calculation to exhibit that the hydrogen spectrum (including the Lamb shift) is contained in our field equations. We will then calculate the Lamb shift $\Delta(n'S_{\frac{1}{2}} - n'P_{\frac{1}{2}})$ for $n = 2, 3, 4$ and determine the magnitude of the fundamental length g_M.

7·1. *The linearization of the field equations.* – Using the same notation as that of the preceding section, the coupled field equations for hydrogen are expressed as follows:

$$\{\gamma_\mu \partial_\mu - e\mathscr{I}^{(p)} + m\}\ \psi^{(e)} = 0\,, \tag{7.1a}$$

$$\{\gamma_\mu \partial_\mu + e\mathscr{I}^{(e)} + M\}\ \psi^{(p)} = 0\,, \tag{7.1b}$$

where m and M are respectively the inertial masses of the electron and the proton and where the operator $\mathscr{I}$ is defined according to eq. (6.3). (The superscripts (e) and (p) refer to functions of the electron and proton solutions $\psi^{(e)}$ and $\psi^{(p)}$).

The approximation used by Dirac in going from his covariant wave equation to the wave equation for hydrogen was to assume that the proton had infinite mass. This in turn implied that the space parts of the four-potential shall vanish, leaving the time component $A_0 = ie/r$. Adopting this same approximation in our field theory, it is expected, on physical grounds, that as M becomes infinitely large, the space parts of the proton velocity field $\bar{\psi}^{(p)}\gamma_k\psi^{(p)}$ shall vanish, while the remaining time component $\bar{\psi}^{(p)}\gamma_0\psi^{(p)}$ should lead to the Coulomb potential in the corresponding part of the Hamiltonian.

Guided by this physical argument, we consider a solution of (7.1b) to have the limiting form

$$\psi^{(p)} = \exp[-iMt]f(\boldsymbol{r})\begin{pmatrix}1\\0\\0\\0\end{pmatrix}. \tag{7.2}$$

Dividing (7.1b) by M, we have

$$\left\{\frac{1}{M}\gamma_0\partial_0 + \frac{1}{M}\gamma_k\partial_k + \frac{e}{M}\mathscr{A}^{(\ )} + 1\right\}\psi^{(p)} = 0\,. \tag{7.3}$$

Clearly, as M approaches infinity, (7.3) takes the form

$$\left\{\frac{1}{M}\gamma_0\partial_0 + 1\right\}\exp[-iMt]f(\boldsymbol{r})\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = 0\,, \tag{7.4}$$

and (7.2) correspondingly approaches an exact solution.

Next, it is observed that the space part $f(\boldsymbol{r})$ of $\psi^{(p)}$ may be chosen arbitrarily, still maintaining the validity of (7.4). Once again, appealing to physical arguments, it is observed that taking

$$f(\boldsymbol{r}) = \sqrt{\delta(\boldsymbol{r})}\,, \tag{7.5}$$

leads to the appearance of the Coulomb potential in the electron eq. (7.1a). The square root of the δ-function means that a bilinear form of the proton wave function is proportional to the Dirac δ-function. This is, in fact, the normalization coefficient for the proton wave function for the infinite mass case, and is another reason for choosing the functional form (7.5).

With (7.2) and (7.5), we have

$$\psi^{(p)}\gamma_k\psi^{(p)} = 0\,, \qquad \psi^{(p)}\gamma_0\psi^{(p)} = \delta(\boldsymbol{r})\,, \qquad (k = 1, 2, 3). \tag{7.6}$$

As indicated in the Appendix, we now break up $\mathscr{I}$ into three parts

$$\mathscr{I} = \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3 .$$

Inserting (7.6) into (A.8), the interaction operator $\mathscr{I}_1^{(p)}$ takes the form

$$\mathscr{I}_1^{(p)} = \gamma_0 e/r . \tag{7.7}$$

This is the conventional Coulomb potential term which appears in the Dirac equation for hydrogen.

In a similar fashion, the substitution of (7.6) into (A.13) gives the contribution

$$\mathscr{I}_2^{(p)} = e\xi\gamma \cdot \int_{\mathrm{i}} \frac{\boldsymbol{r} \times \mathrm{d}\boldsymbol{r}}{r^3} . \tag{7.8}$$

This is just the added contribution coming from the pseudovector part of the electromagnetic four-potential in the generalized Dirac theory which was derived in an earlier publication ([9]). The solution of the Dirac equation for hydrogen with both $\mathscr{I}_1^{(p)}$ and $\mathscr{I}_2^{(p)}$ present in the Hamiltonian, was carried out exactly by the authors ([12]). It was found that the contribution of $\mathscr{I}_2^{(p)}$ to the measured Lamb shift is indeed small and contributes to this value only because of the presence of an external magnetic field in the experiment. It was assumed in this preceding work that the major portion of the Lamb shift was due to the radiative effects which arise in the conventional theory of quantum electrodynamics. The present theory, however, does not involve any such radiative effects and therefore the explanation for the Lamb shift must reside primarily in $\mathscr{I}_3^{(p)}$ (eq. (A.18)) which depends on the constant length g_M.

Before $\mathscr{I}_3^{(p)}$ can be determined, the two two-component spinors q_α must be calculated.

Inserting (7.6) into (3.4), we have

$$\begin{cases} \sigma_\mu \partial_\mu \varphi_1^{(p)} = -\, 4\pi i e\, \delta(\boldsymbol{r}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \sigma_\mu \partial_\mu \varphi_2^{(p)} = 4\pi i e\, \delta(\boldsymbol{r}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \end{cases} \tag{7.9}$$

The solution of these equations is carried out by the Fourier transform

([12]) M. SACHS and S. L. SCHWEBEL: *Ann. of Phys.*, **8**, 475 (1959).

method [13]; giving

$$(7.10)\qquad \left|\begin{array}{l} \varphi_1 = -\dfrac{ie}{r^3}\begin{pmatrix} x_3 \\ x_1 + ix_2 \end{pmatrix}, \\ \varphi_2 = \dfrac{ie}{r^3}\begin{pmatrix} x_1 - ix_2 \\ -x_3 \end{pmatrix}. \end{array}\right.$$

Finally, substituting (7.10) into (A.17) gives (using $a_1 = -a_2 = +1$) (see Section (6·1))

$$(7.11)\qquad \mathscr{I}_3^{(p)} = 16\pi i g_M \frac{e}{r^3}(\boldsymbol{r}\times\boldsymbol{\gamma})_3 .$$

Combining (7.7), (7.8) and (7.11), the linearized Dirac eq. (7.1a) for hydrogen can be written in the form

$$(7.12)\qquad \left\{-\alpha_r \hat{p}_r - i\hbar\frac{\alpha_r}{r}\beta\hat{K} + \frac{e^2}{r} + (16\pi g_M e^2)\frac{(\boldsymbol{r}\times\boldsymbol{\alpha})_3}{r^3} - \beta m + E\right\}\psi = 0 ,$$

where the notation used is

$$\beta \equiv \gamma_0 , \qquad \boldsymbol{\alpha} = i\gamma_0\boldsymbol{\gamma} , \qquad \hbar\hat{K} = \beta\{\boldsymbol{\sigma}\cdot(\boldsymbol{L} - e\boldsymbol{r}\times\boldsymbol{B}) + \hbar\} , \qquad \psi \equiv \psi^{(e)}$$

and

$$(7.13)\qquad \boldsymbol{B} = -\xi e\int_1 \frac{\boldsymbol{r}\times d\boldsymbol{r}}{r^3} , \qquad \hat{p}_r = -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right), \qquad \alpha_r = \frac{\boldsymbol{\alpha}\cdot\boldsymbol{r}}{r} .$$

Following the usual procedure, we make the transformation

$$\varrho = \eta r ,$$

where

$$\eta = +\sqrt{\eta_1\eta_2} , \qquad \eta_1 = m + E , \qquad \eta_2 = m - E .$$

The wave eq. (7.12) then takes the form

$$(7.14)\qquad \hat{\mathscr{H}}\psi \equiv (\hat{\mathscr{H}}_0 + \hat{V})\psi = -i\frac{E}{\eta}\psi ,$$

where

$$(7.15)\qquad i\eta\hat{\mathscr{H}}_0 = i\left\{\alpha_\varrho\left(\frac{\partial}{\partial\varrho} + \frac{1}{\varrho}\right) - \frac{\alpha_\varrho\beta\hat{K}}{\varrho} - i\left|\frac{\gamma}{\varrho} + \frac{\beta m}{\eta}\right|\right\}\eta ,$$

[13] The details of these calculations are contained in an unpublished report by the authors, prepared at San Jose State College, San Jose, California.

is the unperturbed (generalized) Dirac Hamiltonian for hydrogen ([12]). The symbol γ will be used to denote the fine structure constant, $(e^2/\hbar c)$. Making use of our previous results ([12]), the eigenvalues of $i\hat{\mathscr{H}}_0$ are

$$\frac{E_{T_n}}{\eta} = \frac{m}{\eta}\left[1 + \frac{\gamma^2}{(s+n)^2}\right]^{-\frac{1}{2}}, \tag{7.16}$$

with

$$s^2 = K^2 - \gamma^2$$

$$K^2 = (T + \tfrac{1}{2})^2 - (3\xi\gamma)^2$$

and T is the half-integer angular momentum quantum number. (It is recalled that as $\xi \to 0$, $T \to J$, $\hat{K} \to \hat{k}$ of the ordinary Dirac theory.)

The eigenvalues of $\hat{\mathscr{H}}_0$ were found ([12]) to maintain the accidental degeneracy of the excited states of hydrogen, even with $\xi \neq 0$. However, the perturbing term (in dimensionless form)

$$i\hat{V} = \frac{\varkappa}{\varrho^2}(\boldsymbol{\rho}\cdot\boldsymbol{\alpha})_\varrho, \tag{7.17}$$

lifts this degeneracy. The strength of this term is measured by the constant

$$\varkappa = 16\pi g_M \gamma\eta = \frac{16\pi}{[(s+n)^2+\gamma^2]^{\frac{1}{2}}}\left(\frac{g_M}{\lambda_c}\right)\gamma^2, \tag{7.18}$$

where (inserting $\hbar$, c)

$$\lambda_c = \hbar/mc$$

is the electron Compton wave length.

It is observed that $\hat{V}$ diverges at the origin as ϱ^{-2} while the ordinary Coulomb term diverges only as ϱ^{-1}. In order to ensure that the solutions of the unperturbed equation have the proper behavior at the origin to justify a rapidly convergent perturbation expansion of operators which depend on ϱ^{-n} $(n \geqslant 2)$, we redefine the zero order and the perturbing energy operator by adding and substracting the term

$$\frac{i\varkappa\alpha_\varrho}{\varrho^2}$$

to the Hamiltonian. Thus, we take

$$i\hat{\mathscr{H}}_0' = i\left(\hat{\mathscr{H}}_0 - \frac{\varkappa\alpha_\varrho}{\varrho^2}\right), \tag{7.19}$$

as the unperturbed Hamiltonian operator. The remaining part of $i\hat{\mathscr{H}}$,

$$i\hat{V}' = \frac{\varkappa}{\varrho^2}\left|\frac{(\boldsymbol{\rho}\times\boldsymbol{\alpha})_3}{\varrho} + i\alpha_\varrho\right|, \tag{7.20}$$

is now considered as the perturbing term. Clearly

$$\hat{\mathscr{H}} = \hat{\mathscr{H}}_0 + \hat{V}' = \hat{\mathscr{H}} - \hat{V}'$$

The wave equation

$$\hat{\mathscr{H}}_0'\psi = -\frac{i}{\eta}E\psi ,$$

can be solved exactly giving

$$\psi = \psi_D \exp[-\varkappa/\varrho] \tag{7.21}$$

where ψ_D are the eigensolutions of the original Dirac Hamiltonian $\hat{\mathscr{H}}_0$, (eq. (7.15)).

The perturbation operator $i\hat{V}'$ (7.20) has the form

$$i\hat{V}' = \frac{\varkappa}{\varrho^2}\left\{\sin\theta\exp[i\varphi]\begin{pmatrix}0 & \Lambda\\ \Lambda & 0\end{pmatrix} + i\cos\theta\begin{pmatrix}0 & \sigma_3\\ \sigma_3 & 0\end{pmatrix}\right\}, \tag{7.22}$$

where

$$\Lambda = 2i\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \tag{7.23}$$

and

$$\sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

is the usual Pauli matrix.

7·2. *The Lamb shift.* – The double (accidental) degeneracy of the first excited state of hydrogen is described by the two orthogonal wave vectors $|+|K|\rangle$ and $|-|K|\rangle$, corresponding to the same energy state (K is defined in eq. (7.16)). Since the alteration of the wave function due to $\xi \neq 0$ is very slight (its influence on the calculated Lamb shift for the *isolated* atom is beyond the experimental accuracy of the measured value) [12] we will take these eigenvectors proportional to the ordinary Dirac solutions [14] for $j = \frac{1}{2}$, $j_3 = \pm\frac{1}{2}$,

[14] See, for example, H. A. Bethe and E. E. Salpeter: *Quantum Mechanics of One- and Two-Electron Atoms* (New York, 1957).

$n=1, 2, 3$. Using (7.21), they are, for $j_3=+\frac{1}{2}$ (*),

$$|-|K|\rangle_n=\frac{(\eta/\varrho)}{\sqrt{4\pi}}\begin{pmatrix} F_-(n) \\ 0 \\ -i\cos\theta G_-(n) \\ -i\sin\theta\exp[i\varphi]G_-(n) \end{pmatrix}\exp[-\varkappa/\varrho]\,, \tag{7.24}$$

$$|+\ K\ \rangle_n=\frac{(\eta/\varrho)}{\sqrt{4\pi}}\begin{pmatrix} \cos\theta F_+(n) \\ \sin\theta\exp[i\varphi]F_+(n) \\ -iG_+(n) \\ 0 \end{pmatrix}\exp[-\varkappa/\varrho]\,, \tag{7.25}$$

where $|K|=1$. $\eta F_\pm/\varrho$ and $\eta G_\pm/\varrho$ are the large and small components of the radial part of the Dirac solution, with the appropriate $\pm K$ values.

It is readily verified that after the integration over angular variables has been carried out,

$$\langle\pm|K||i\hat{V}'|\mp|K|\rangle=0 \tag{7.26}$$

and

$$\langle\pm|K||iV'|\pm|K|\rangle=\pm\frac{4}{3}\frac{\tau}{\eta}\int_0^\infty\frac{F_\pm G_\pm}{\varrho^2}\exp[-2\varkappa/\varrho]\,\mathrm{d}\varrho\,, \tag{7.27}$$

where, inserting $\hbar$, c

$$\tau_n=\eta\hbar c\varkappa=\frac{16\pi}{(s+n)^2+\gamma^2}\left(\frac{g_M}{\lambda_c}\right)\gamma^3(mc^2)\,. \tag{7.28}$$

Thus, the energy splitting between the two states, conventionally labelled $(n+1)S_{\frac{1}{2}}$ and $(n+1)P_{\frac{1}{2}}$, is

$$\Delta_n^{(\mathrm{H})}=\tfrac{4}{3}\tau_n|I_+ + I_-|_n\,, \tag{7.29}$$

where

$$|I_\pm|_n=\eta^{-1}\int_0^\infty\frac{F_\pm(n)\,G_+(n)}{\varrho^2}\exp[-2\varkappa/\varrho]\,\mathrm{d}\varrho\,. \tag{7.30}$$

Evaluating $I_\pm$ ([13]) for $n=1, 2, 3$ (corresponding to the principal quantum

(*) The same results obtained below are obtained as expected for the corresponding states with $j_3=-\frac{1}{2}$:

numbers 2, 3, 4) we find (to the accuracy desired in this calculation)

$$|I_+ + I_-|_{n=1} = \gamma/3 \,, \tag{7.31a}$$

$$|I_+ + I_-|_2 = 2\gamma/9 \,, \tag{7.31b}$$

$$|I_+ + I_-|_3 = \gamma/6 \,. \tag{7.31c}$$

Combining (7.28), (7.29) and (7.31) we find that to order γ^4

$$\Delta_1^{(\mathrm{H})} = (16\pi/9)\gamma^4(g_M/\lambda_c)(mc^2) \,, \tag{7.32a}$$

$$(\Delta_2^{(\mathrm{H})}/\Delta_1^{(\mathrm{H})}) = 8/27 \qquad (= 0.296 \ldots) \,, \tag{7.32b}$$

$$(\Delta_3^{(\mathrm{H})}/\Delta_1^{(\mathrm{H})}) = 1/8 \qquad (= 0.125) \,. \tag{7.32c}$$

Experimentally, [15,16,4]

$$(\Delta_3^{(\mathrm{H})}/\Delta_1^{(\mathrm{H})})_{\mathrm{exp}} = 315.0/1057.77 = .298 \,. \tag{7.33}$$

The significance of this ratio is that it is independent of the magnitude of the one extra fundamental constant g_M that appears in our theory. The theoretical value for this ratio according to the self-consistent field theory presented in this paper is therefore within .7 % of the experimental value.

If we now substitute the experimental value for the Lamb shift [15]

$$\Delta_1^{(\mathrm{H})} = (1.057\,77 \pm 10^{-4}) \cdot 10^9 \, h \ \mathrm{erg} \,,$$

into (7.32) we find that the fundamental length g_M has the value

$$g_M = (2.087 \pm .001) \cdot 10^{-14} \ \mathrm{cm} \,. \tag{7.34}$$

Thus far we have seen that the theory predicts the two Lamb shift measurements for the hydrogen energy levels (with principal quantum numbers 2 and 3) to a quite high degree of precision, and that these results are compatible with the magnitude (7.34) of the *one* extra constant g_M inherent in our formalism.

The Lamb shift

$$\Delta_1^{(\mathrm{H})}(4S_{\frac{1}{2}} - 4P_{\frac{1}{2}})$$

corresponding to that state of hydrogen with $n = 3$ has not yet been measured. We have calculated this quantity in order to compare it with the result of the conventional calculation in terms of radiative effects.

[15] S. Triebwasser, E. S. Dayhoff and W. E. Lamb jr.: *Phys. Rev.*, **89**, 98 (1953).
[16] W. E. Lamb jr. and T. M. Sanders jr.: *Phys. Rev.*, **103**, 313 (1956).

With (7.32c) and (7.34) we have

$$\Delta_3^{(\mathrm{H})}/h = 132.22 \text{ MHz} .$$

The result obtained with the conventional theory is [4]

$$\Delta_3^{(\mathrm{H})}/h = (133.10 \pm .02) \text{ MHz} .$$

Thus, while both theories seem to give the same value for the Lamb shift in the states with $n=1$ and $n=2$, they apparently differ by the order of 1 out of 133 MHz in the state with $n=3$. This would then indicate that if our linearization procedure is good to .01, an investigation of the Lamb shift in the higher principle quantum number states would indeed be significant.

It is also quite interesting to note that our theory and the conventional one, while being so different with respect to both formalism and interpretation should give results for the Lamb shift which are so close. At this stage of our calculation, we can only conjecture that this is not pure coincidence.

In view of our self-consistent theory, the Lamb shift for D and He^+ should be determined respectively from sets of three and five coupled non-linear equations because of the number of particles in their respective nuclei. To treat these atoms as we did hydrogen would be to assume that their nuclei were a single particle with increased mass and charge (for He^+). It is not clear at this stage that such an assumption is an accurate one. If, however, such an assumption were made, then in the approximation made thus far in this section, D would have the same Lamb shift as H (since Δ in eq. (7.29) does not depend on the nuclear mass). Similarly He^+ would have Lamb shift 16 times greater than hydrogen (for the $n=2$ state) since $\Delta \alpha \gamma^4$ and in He^+, γ^4 must be replaced by $(2\gamma)^4$. The experimental results are [4]

$$\frac{\Delta_1^{(\mathrm{D})}}{\Delta_1^{(\mathrm{H})}} \simeq 1.001 \qquad \text{and} \qquad \frac{\Delta_1^{(\mathrm{He}^+)}}{\Delta_1^{(\mathrm{H})}} \simeq 14 .$$

8. – Discussion.

An extension of the present work would be a study of the other states of positronium and of systems composed of particles of different masses. Such a study would indicate whether additional terms have to be added to the interaction operator $\mathscr{I}(e)$, since the only property of this operator needed to determine the annihilation state of positronium is

(8.1) $$\mathscr{I}(e)\psi(e) = 0 .$$

For example, if in $\mathscr{L}_L$ (eq. (4.6)), $\bar{\psi}\gamma_\mu\psi$ is replaced by $\bar{\psi}\gamma_\mu(a+b\gamma_5)\psi$, then eq. (8.1) is satisfied for the same solution (6.10). Although $\mathscr{L}_L$ was originally constructed to have a counterpart in classical electromagnetic theory, such additional terms, without a classical counterpart, may be needed to provide a complete description of the electromagnetic *elementary interaction.*

As mentioned in Section **7**, a study of the fine structure of He^+ and D should be a very significant one, especially since the energy level structure of these atoms, including the Lamb shift, is known to high precision. Such a study would begin, in view of our theory, with an investigation of the non-linear equations, coupling the neutron-proton and electron to form D and of the four-nucleon α-particle coupled to the electron to form He^+. The respective field equations for these systems would be three and five coupled non-linear equations of the form (4.10), extended to

$$\hat{O}(i,j)\psi^{(j)} = 0\,, \qquad i \neq j = \begin{cases} 1,2,3 & \text{for D,} \\ 1,2,3,4,5 & \text{for } He^+. \end{cases} \tag{8.2}$$

Also, other interactions besides those considered in the explicit expression (4.11) would have to be considered in (8.2) to account for the nuclear coupling of the nucleons of these many particle systems.

With regard to nuclear forces, the present theory should be extended to include interactions which correspond to the existence of the strong, short-range interaction observed in high energy nuclear experiments. It should be noted, however, that the various elementary particles, with their associated, masses, spins, modes and rates of decay would be interpreted in our theory not as properties of each of the constituent particles of a system but rather in terms of various modes of interaction between a system of particles.

One of the implications of the theory discussed in this paper which could possibly be subject to experimental corroboration by more precise measurements is the denial of the spontaneous decay process. The theory predicts that if a system is in a particular state, it will remain in this state unless another system enters, causing a transfer of energy to or from the latter. Thus, for example, the present theory implies that the decay of an excited nucleus occurs only because of its interaction with its environment. It would then appear that the « half-life » of a nucleus in an excited state is not truly a constant associated only with that nucleus. It is clear, however, that measurements of deviations from the constancy of the half-life of the excited nucleus should be extremely small due to the relatively small reaction which the *macroscopic* observer has when observing the *microscopic* nucleus in its environment. These qualitative considerations have not yet been applied quantitatively.

The interpretation of the Maxwell field equations as a covariant representation of the dynamics of charged particles in terms of field variables is a fun-

damental tenet of our theory. The role played by the electric charge e of the classical theory enters here as the coupling constant e^2, measuring the strength of the electromagnetic interaction. In view of this approach, the appearance of the inertial mass m in $\mathscr{L}_{\text{D}}$ (eq. (4.4)) is an anomaly, for it would be expected that m as well as e should play a role in the elementary interaction of the unified field theory ([17]). Hence, the present theory would require that the Einstein field equations be interpreted as a covariant representation of gravitational matter in terms of field variables. Thus, in the unified field theory, the inertial mass would not appear in $\mathscr{L}_{\text{D}}$, but rather in a coupling constant as a measure of the strength of the elementary interaction. The latter would be expressed in terms of the Einstein, Maxwell, and Dirac field variables. The authors are currently investigating this aspect of the theory.

Appendix

The annihilation state.

The Maxwell field eq. (3.4), with the source terms given by eq. (6.12), are

$$\sigma_\mu \partial_\mu \varphi_1 = -8\pi i e \begin{pmatrix} 1 \\ -\exp[2imt] \end{pmatrix} \equiv \Upsilon_1 , \tag{A.1a}$$

$$\sigma_\mu \partial_\mu \varphi_2 = 8\pi i e \begin{pmatrix} -\exp[-2imt] \\ 1 \end{pmatrix} \equiv \Upsilon_2 . \tag{A.1b}$$

We proceed to solve these equations by using the Fourier transform method. Let

$$\varphi(\boldsymbol{x}) = \int \varphi(\boldsymbol{k}) \exp[ik_\mu x_\mu] \, \mathrm{d}^4 k . \tag{A.2}$$

Carrying out the indicated operations, we find that

$$\varphi_1(\boldsymbol{k}) = -\frac{i\tilde{\sigma}_\beta k_\beta e}{(2\pi)^4 k_\alpha^2} \int \exp[-ik_\mu x_\mu](-8\pi i) \begin{pmatrix} 1 \\ -\exp[2imt] \end{pmatrix} \mathrm{d}^4 x , \tag{A.3a}$$

and

$$\varphi_2(\boldsymbol{k}) = -\frac{i\tilde{\sigma}_\beta k_\beta e}{(2\pi)^4 k_\alpha^2} \int \exp[-ik_\mu x_\mu] (8\pi i) \begin{pmatrix} -\exp[-2imt] \\ 1 \end{pmatrix} \mathrm{d}^4 x , \tag{A.3b}$$

([17]) For the solution of the two-body problem of gravitational interaction, according to the general theory of relativity, see A. Einstein and L. Infeld: *Can. Journ. Math.* **1**, 209 (1949); A. Einstein, L. Infeld and B. Hoffmann: *Ann. Math.*, **39**, 66 (1938). A. Einstein and L. Infeld: *Ann. Math.*, **41**, 797 (1940); L. Infeld: *Rev. Mod. Phys.*, **29**, 398 (1957).

where

$$\tilde{\sigma} \equiv \{iI; -\sigma_k\} .$$

The requirement of our theory that the field variable must vanish when there are no sources leads to a complication which is resolved below.

From eq. (A.3) we obtain the following solutions:

$$\text{(A.4}a\text{)} \qquad \varphi_1(\boldsymbol{x}) = -\frac{8\pi ie}{(2\pi)^4}\int \exp\left[ik_\mu(x_\mu - x'_\mu)\right] \frac{(-i\tilde{\sigma}_\beta k_\beta)}{k_\lambda^2} \begin{pmatrix} 1 \\ -\exp\left[2imt'\right] \end{pmatrix} \mathrm{d}^4k\, \mathrm{d}^4x',$$

$$\text{(A.4}b\text{)} \qquad \varphi_2(\boldsymbol{x}) = \frac{8\pi ie}{(2\pi)^4}\int \exp\left[ik_\mu(x_\mu - x'_\mu)\right] \frac{(-i\tilde{\sigma}_\beta k_\beta)}{k_\lambda^2} \begin{pmatrix} -\exp\left[-2imt'\right] \\ 1 \end{pmatrix} \mathrm{d}^4k\, \mathrm{d}^4x' .$$

Carrying out these integrations, we find that

$$\text{(A.5}a\text{)} \qquad \varphi_1(\boldsymbol{x}) = \frac{4\pi e}{m} \begin{pmatrix} 0 \\ \exp\left[2imt\right] \end{pmatrix},$$

$$\text{(A.5}b\text{)} \qquad \varphi_2(\boldsymbol{x}) = \frac{4\pi e}{m} \begin{pmatrix} \exp\left[-2imt\right] \\ 0 \end{pmatrix}.$$

The solutions (A.5) differ from (A.4) in that the latter satisfy the differential eq. (A.1) but the former do not. The apparent difficulty is associated with the treatment of the constant part of the source term in solutions (A.4). To obtain a consistent treatment of the singularities, we appeal to the invariance properties of the spinors involved.

In general, the Maxwell field eq. (3.2) can be written as follows ([6]):

$$\text{(A.6)} \qquad \sigma_\mu \partial_\mu \begin{pmatrix} \varphi_1(1) & \varphi_2(1) \\ \varphi_1(2) & \varphi_2(2) \end{pmatrix} = \begin{pmatrix} \Upsilon_1(1) & \Upsilon_2(1) \\ \Upsilon_1(2) & \Upsilon_2(2) \end{pmatrix}.$$

The determinant of the matrix on the right-hand side of this equation is the invariant (according to the spinor algebra) ([6])

$$\Upsilon_1^{\mathrm{tr}}(i\sigma_2)\,\Upsilon_2 .$$

For the particular source terms given in eq. (A.1), this invariant is zero. This imposes the same value on the matrix of the field variables, which is the invariant

$$\varphi_1^{\mathrm{tr}}(i\sigma_2)\varphi_2 .$$

A consequence of these considerations is an alteration of the solutions (A.5) to

$$\text{(A.7}a\text{)} \qquad \varphi_1(\boldsymbol{x}) = \frac{4\pi e}{m} \begin{pmatrix} 1 \\ \exp\left[2imt\right] \end{pmatrix},$$

$$\text{(A.7I)} \qquad \varphi_2(\boldsymbol{x}) = \frac{4\pi e}{m} \begin{pmatrix} \exp\left[-2imt\right] \\ 1 \end{pmatrix},$$

which are the solutions (6.19). Thus, we define the solutions in such a way that the integral form (A.4) is to be used when derivatives are to be calculated; otherwise, the functions are given by eq. (A.7).

A similar situation occurs with the velocity field coupling terms in $\mathscr{L}_L$: The scalar term gives the following contribution to $\mathscr{I}$:

$$\mathscr{I}_1 \equiv e\gamma_\nu \int \bar{\psi}\gamma_\nu\psi\, S(\boldsymbol{x}-\boldsymbol{x}')\,\mathrm{d}^4x', \tag{A.8}$$

where $S(\boldsymbol{x}-\boldsymbol{x}')$ (eq. (4.7)) can be written in the factored form

$$S(\boldsymbol{x}-\boldsymbol{x}') = \int \mathrm{d}^4x''\,\mathrm{d}^4k' \begin{pmatrix} i\sigma_\alpha k'_\alpha \\ (2\pi)^4 k'^2_0 \end{pmatrix} \cdot \exp\left[ik'_\mu(x_\mu - x''_\mu)\right] \mathrm{d}^4x'\,\mathrm{d}^4k \left(-\frac{i\tilde{\sigma}_\beta k_\beta}{(2\pi)^4 k'_\varepsilon}\right) \exp\left[ik_\mu(x''_\mu - x'_\mu)\right]. \tag{A.9}$$

A comparison of this equation with the kernel of eq. (A.4) shows $S(\boldsymbol{x}-\boldsymbol{x}')$ to be equivalent to a double application of that kernel. Thus, in the event that any of the functions $\bar{\psi}\gamma_\nu\psi$ in eq. (A.8) are constant, the procedure outlined above for φ_α is doubly followed here. Carrying out this procedure for our specific solution (6.10), we obtain for (A.8),

$$\mathscr{I}_1 = \frac{e}{2m^2}\left\{\gamma_0 - i\gamma_1\cos 2mt - i\gamma_2 \sin 2mt\right\}. \tag{A.10}$$

It is readily verified that for ψ given by eq. (6.10),

$$(i\gamma_1\cos 2mt + i\gamma_2 \sin 2mt)\psi = \gamma_0\psi. \tag{A.11}$$

Thus, from eq. (A.10) and (A.11),

$$\mathscr{I}_1\psi = 0. \tag{A.12}$$

The pseudoscalar velocity coupling term in $\mathscr{L}_L$ contributes to the Dirac equation

$$\mathscr{I}_2 = \left\{-\frac{i\xi e}{4}\,\varepsilon_{\mu\nu\lambda\varrho}\gamma_\nu \int \bar{\psi}\gamma_\lambda\psi\,\mathrm{d}^4x' \int \partial_\varrho S(\boldsymbol{x}-\boldsymbol{x}')\,\mathrm{d}x_\mu\right\}. \tag{A.13}$$

To evaluate this term, we consider the indefinite integral

$$\int \partial_\varrho S(\boldsymbol{x}-\boldsymbol{x}')\,\mathrm{d}x_\mu = \frac{1}{(2\pi)^4}\int\int \mathrm{d}x_\mu\,\partial_\varrho \frac{\exp\left[ik_\alpha(x_\alpha - x'_\alpha)\right]}{k^2_\beta}\,\mathrm{d}^4k = \tag{A.14}$$

$$= \frac{1}{(2\pi)^4}\int \frac{k_\varrho}{k_\mu}\,\frac{\exp\left[ik_\alpha(x_\alpha - x'_\alpha)\right]}{k^2_\beta}\,\mathrm{d}^4k.$$

For the solution ψ given by eq. (6.10), $\bar{\psi}\gamma_\lambda\psi$ is either a constant or a function of x_0. Therefore, the integration over the space variables can be carried out. We find that

$$\mathscr{I}_2 = -\frac{i\xi e}{4}\varepsilon_{\mu\nu\lambda\varrho}\gamma_\nu\int\bar{\psi}\gamma_\lambda\psi\,\mathrm{d}x_0\int\frac{\mathrm{d}^4k}{k_\alpha^2}\,\frac{1}{(2\pi)^4}\,\frac{k_\varrho}{k_\mu}\exp[ik_0(x_0-x_0')]\exp[i\boldsymbol{k}\cdot\boldsymbol{r}]\cdot \cdot\delta(k_1)\,\delta(k_2)\,\delta(k_3)\,. \tag{A.15}$$

Since ϱ cannot be equal to μ, it is clear that the function $(k_\varrho/k_\mu)\delta(k_1)\delta(k_2)\delta(k_3)$ eq. (A.15) is an odd function in one of the k_α components ($\alpha=1, 2, 3$). Therefore,

$$\mathscr{I}_2\psi = 0\,. \tag{A.16}$$

Finally, we evaluate the interaction term

$$\mathscr{I}_3\psi = -ig_M\sum_{\alpha=1}^{2}a_\alpha(\varphi_\alpha^\dagger\cdot\Gamma_\alpha-\gamma_0\Gamma_\alpha^\dagger\gamma_0\cdot\varphi_\alpha)\psi\,. \tag{A.17}$$

Substituting the solutions (A.7) into (A.17) and using (A.11), we find that

$$\mathscr{I}_3\psi = -32\pi^2\left(\frac{e}{m}\right)g_M(a_1+a_2)\gamma_5\psi\,. \tag{A.18}$$

Choosing $a_1=-a_2$, we have

$$\mathscr{I}_3\psi = 0\,.$$

Thus,

$$(\mathscr{I}_1+\mathscr{I}_2+\mathscr{I}_3)\psi = \mathscr{I}\psi = 0\,. \tag{A.19}$$

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